Noise Leakage Suppression in Multivariate FRF Measurements Using Periodic Excitations

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Abstract

Due to the non-periodic nature of noise, the steady state response of a dynamic system to a periodic input is still subject to noise transients (noise leakage errors). For lightly damped systems these noise transients (significantly) increase the variance of frequency response function (FRF) measurements [1]. This paper presents a method for suppressing the noise transients in FRF measurements using periodic excitations. It is based on a local polynomial approximation of the noise leakage error and is an extension of the results of [1] to multivariable systems. Compared with the local polynomial method for random excitations [2, 3], no local polynomial approximation of the frequency response matrix is made. Irrespective of the number of inputs and outputs, it is shown in this paper that 2 periods of the state state response are enough to suppress the noise transients and to estimate the input-output noise covariance matrix. Since no distinction can be made between the system and noise transients, the presented method is also applicable to the first 2 periods of the transient response of the system to a periodic input. For lightly damped systems this results in a significant reduction of the measurement time.

1 Introduction

Frequency response function (FRF) measurements give quickly insight in the dynamical behaviour of complex systems [4, 5]. They are very useful for constructing and/or validating parametric transfer function model approximations of real life systems [6, 7, 8]. These parametric transfer function models are then used for virtual prototyping of new products, for physical interpretation and/or better understanding of the underlying physical phenomena, for prediction and/or control, for monitoring and fault detection . . .

The frequency response matrix can be measured using arbitrary [4, 5] or periodic [9, 10] excitations. Arbitrary excitations and the related algorithms have the following advantages: (i) a larger frequency resolution of the FRM measurement, and (ii) the fact that operational input-output data can be handled. The disadvantages are (i) the presence of system transients (leakage errors) in the FRM estimate, (ii) the lack of distinction between noise and nonlinear distortions for stationary inputs, and (iii) uncorrelated input signals have a non-diagonal finite sample power spectrum matrix (an infinite amount of data samples is needed to obtain a diagonal sample power spectrum matrix for the input signals). Note that the leakage problem is reduced by windowing techniques [4, 5] or a local polynomial approximation of the FRM and the transients [2, 3]. Periodic excitations have the following advantages: (i) in steady state the system transients (leakage errors) are zero, (ii) the noise can be separated from the nonlinear distortions [11, 12, 13, 14, 15], and (iii) uncorrelated input signals can be constructed for finite samples. The disadvantages are (i) the smaller frequency
resolution, and (ii) the fact that no operational data can be handled. This paper studies FRM measurements using periodic excitations.

Recently [1] a method has been developed to suppress nonparametrically the noise (and system) transients (leakage errors) in frequency response function (FRF) and noise (co-)variance estimates of single-input, single output systems excited by periodic signals. This paper extends the results of [1] to multiple-input, multiple-output systems where all inputs and outputs are disturbed by noise (= errors-in-variables framework). For lightly damped systems, the proposed method either reduces significantly the experiment duration (measuring 2 periods is enough) or, for a given measurement time, increases significantly the frequency resolution of the FRF measurement. If the noise (and/or system) transients are the dominant error source, then the proposed method also reduces significantly the covariance matrix of the FRF estimates.

To separate the noise from the nonlinear distortions in the FRM measurement of a nonlinear system, one needs a number of FRM measurements with independent random realisations of Gaussian-like input signals [11, 12, 13, 14]. Irrespective of the number of inputs \( n_u \) and outputs \( n_y \), it is shown in this paper that 2 independent realisations are enough. For lightly damped systems this results again in either a significant reduction of the measurement time or, for a given measurement time, a significant increase in frequency resolution of the FRF measurement.

Summarised the main contributions of the paper are

- The reduction of the number of signal periods that are necessary for the noise analysis: instead of at least \( n_y + 5 \) signal periods in steady state [8, 16], 2 signal periods of the transient response are sufficient.
- The reduction of the number of experiments with independent random realisations of the input for detecting the nonlinear distortions: instead of at least \( n_y + 5 \) experiments [8, 16], 2 are sufficient to separate the noise from the nonlinear distortions.

The proposed method reduces the experimental time or, for a given measurement time, it allows to increase the frequency resolution of the FRM measurement. As such, the lower frequency resolution drawback of periodic signals w.r.t. random excitations is weakened considerably.

The outline of the paper is as follows. Section 2 gives an overview of broadband periodic excitations signals that are suitable for measuring the FRM of linear (Section 2.1) and nonlinear (Section 2.2) dynamical systems. These signals are then used for estimating nonparametrically the level of the noise (Section 3.1) and the nonlinear distortions (Section 3.2) on the measured input-output Fourier coefficients via the local polynomial approach. Section 3.4 shows that the proposed method can also handle the system transient. How to calculate the FRM and its uncertainty from this nonparametric information is discussed in Section 4. The theory is illustrated on a real measurement (Section 5) example. Finally, some conclusions are drawn in Section 6.

2 Overview of periodic excitations for multivariate FRF measurements

In this section we discuss the pros and cons of different periodic excitation signals for measuring the frequency response matrix (FRM) of a multivariate dynamical system. We first handle the linear case, and next highlight the peculiarities of FRM measurements in the presence of nonlinear distortions.

2.1 Linear dynamical systems

2.1.1 General

Consider a linear dynamical system with \( n_u \) inputs \( u(t) \) and \( n_y \) outputs \( y(t) \) (see Figure 1). It is excited by a periodic signal \( r(t) \) via a (non)linear actuator. The input-output discrete Fourier transform (DFT) spectra
Figure 1: Linear time-invariant (LTI) system with $n_u$ inputs $u(t)$ and $n_y$ outputs $y(t)$, excited by a periodic $n_u \times 1$ signal $r(t)$ via a (non)linear actuator.

$U(k), Y(k)$ of one period of the steady state response $u(t), y(t)$ to the periodic signal $r(t)$ are related by

$$Y(k) = G(\Omega_k)U(k)$$

(1)

with $G(\Omega_k)$ the $n_y \times n_u$ frequency response matrix (FRM) of the linear time-invariant (LTI) system, $X(k)$ the DFT of the $n_s \times 1$ signal $x(t)$

$$X(k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t) e^{-j\frac{2\pi nt}{N}}$$

(2)

with $N$ the number of time domain samples, and $\Omega_k$ the generalised frequency variable (for continuous-time systems $\Omega_k = j\omega_k$ and $\Omega = s$, and for discrete-time systems $\Omega_k = e^{-j\omega_k T_s}$ and $\Omega = z^{-1}$, with $\omega_k = 2\pi kf_s/N$ and $f_s = 1/T_s$ the sampling frequency). Since $G(\Omega_k)$ contains $n_y n_u$ unknowns, $G(\Omega_k)$ can - in general - not be calculated from the $n_y$ equations (1). In the sequel of Section 2.1, we present a solution to this problem.

2.1.2 Hadamard and orthogonal multisines - multiple experiments

To measure the FRM from $u(t)$ to $y(t)$ in Figure 1 one needs at least $n_u$ experiments with linearly independent input signals $u^{[e]}(t), e = 1, 2, \ldots, n_u$. Collecting (1) for $n_u$ experiments one obtains

$$Y(k) = G(\Omega_k)U(k)$$

(3)

where $X(k) = [X^{[1]}(k) X^{[2]}(k) \ldots X^{[n_u]}(k)]$, with $X = Y, U$, and $U(k)$ a regular $n_u \times n_u$ matrix. The FRM is then found as

$$G(\Omega_k) = Y(k) U^{-1}(k)$$

(4)

To decrease the sensitivity of the FRM estimate (4) to input-output measurement errors, the matrix $U(k)$ should be well conditioned. One can start from one scalar multisine $r_{\text{siso}}(t)$ that excites all frequencies in the band of interest. Next, the $n_u$ linearly independent reference signals are obtained by multiplying the spectrum $R_{\text{siso}}(k)$ with an orthogonal $n_u \times n_u$ matrix $T \ (T^{-1} = T^H)$

$$R(k) = R_{\text{siso}}(k) T$$

(5)

(the $e$th column of $R(k)$ represents the DFT spectrum of the reference signal of the $e$th experiment). With the choice (5), the reference signals of the different experiments are orthogonal to each other for a finite value of the number of samples $N$. In addition, in case of an ideal actuator ($u(t) = r(t)$ in Figure 1), (5) minimises the determinant of the covariance matrix of the FRM estimate [9, 10] (= $D$-optimality).

If the number of inputs is a power of 2, then one can choose $T$ in (5) to be equal to the Hadamard matrix of order $n_u = 2^m$. 
Figure 2: Best linear approximation (BLA) of a nonlinear (NL) period in, same period out (PISPO) multi
variable dynamical system excited by a $n_u \times 1$ random input $u(t)$. The difference $y_s(t)$ between actual output $y(t)$ of the nonlinear system and the output of the BLA is uncorrelated with $u(t)$.

$$T = \frac{1}{\sqrt{n_u}} H_{2u} \quad \text{with} \quad H_{2u} = H_2 \otimes H_{2u-1} \quad \text{and} \quad H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$ (6)

leading to the so-called Hadamard multisines [9] ($\otimes$ stands for the Kronecker product [18]). Note that (6) only requires 1 generator whose output, multiplied with the correct sign, is connected to the different inputs. For any value of the number of inputs, one can choose $T$ in (5) to be equal to the $n_u \times n_u$ DFT matrix

$$T[p,q] = \frac{1}{\sqrt{n_u}} e^{\frac{2\pi i (p-1)(q-1)}{n_u}} \quad \text{with} \quad p, q = 1, 2, \ldots, n_u$$ (7)

leading to the so-called orthogonal multisines [10]. Contrary to the Hadamard solution, $n_u$ generators are needed here.

In quite some applications the power spectra of the operational perturbations differ over the inputs. To cope with this requirement, the matrix in (5) is multiplied by a frequency dependent diagonal $n_u \times n_u$ matrix $D_A(k)$ that defines the shape of the $n_u$ input amplitude spectra

$$R(k) = R_{\text{sISO}}(k) D_A(k) T$$ (8)

where $D_A[p,q](k) = A_p(k) \delta_{pq}$ with $\delta_{pq}$ the Kronecker delta and $A_p(k) \geq 0$. Independent of the choice of the orthogonal matrix $T$, (8) always requires $n_u$ different generators.

### 2.2 Nonlinear dynamical systems

#### 2.2.1 General

Consider a nonlinear multivariate dynamical system whose steady state response to a periodic input has the same period as the input. Assume further that this PISPO (period in, same period out) system is excited by $n_u \times 1$ random input $u(t)$. The actual output of the nonlinear system can then be calculated as the sum of the response of the best (in mean square sense) linear approximation (BLA) of this system and an error term $y_s(t)$. This error term $y_s(t)$ has zero mean (w.r.t. the random realisation of the input), is uncorrelated with the input $u(t)$, and acts as noise on the BLA measurement [8, 19]. Therefore, it is called the stochastic nonlinear distortion. It has been shown in [12, 20] that the BLA of the multivariable nonlinear system is the same within the class of Gaussian-like excitations (Gaussian noise, periodic noise, random phase multisines) with the same Riemann equivalent power spectrum (rms-value and colouring). In the sequel of Section 2.2 we present excitation signals for measuring the BLA and the covariance matrix of the stochastic nonlinear distortions.

#### 2.2.2 Random orthogonal multisines - multiple experiments

To measure the BLA one should ensure the randomness of the error term $y_s(t)$ in Figure 2. Therefore, the nonlinear multivariate system is excited by $n_u$ different random phase multisines $R[p](k)$ [10, 12, 13].
Random phase multisines are periodic signals with a user defined amplitude spectrum $|R_p(k)|$, and a random phase spectrum such that $E\left\{ e^{j\angle R_p(k)} \right\} = 0$, where the phases $\angle R_p(k)$ are independently chosen over the frequency $k$ and the input $p$. Hence, (8) is replaced by

$$R(k) = D_R(k)T$$  (9)

where $D_R[p,q](k) = R_p(k)\delta_{pq}$, and with $T$ defined in (7). Since the stochastic nonlinear distortions $Y_s(k)$ of the $n_u$ experiments with the random orthogonal multisines (9) are not independent of each other, their covariance matrix cannot be estimated [14]. However, their contribution to the BLA can still be measured [10, 12, 13]. This problem is solved by adding the same random phase $\phi_e(k)$ to each input, such that $E\left\{ e^{j\phi_e(k)} \right\} = 0$, where $\phi_e(k)$ is randomly chosen over the frequency $k$ and over the experiment $e$. Hence, (9) is replaced by

$$R(k) = D_R(k)TD\phi(k)$$  (10)

where $D[\phi][p,e](k) = e^{j\phi_p(k)}\delta_{pe}$. Compared with (9), the full random orthogonal multisines (10) (i) allow to estimate the covariance matrix of the stochastic nonlinear distortions $Y_s(k)$, and (ii) decrease the variability of the BLA measurement [14]. Both solutions require $n_u$ generators.

### 3 Nonparametric modelling - the local polynomial method

This section first describes the local polynomial method for suppressing the noise leakage (transient) errors in the sample mean and sample noise covariance matrices of the noisy input-output DFT spectra (Section 3.1). Next, it is shown how to separate the noise from the nonlinear distortion in the measured input-output DFT spectra (Section 3.2). Finally, the influence of system transients (leakage errors) on the estimated sample mean and sample noise covariances is studied (Section 3.4).
Figure 4: DFT spectrum of \( P = 2 \) consecutive periods of a noiseless (left) and a noisy signal (right). The grey arrows represent the noise, and the black arrows the signal.

### 3.1 Linear dynamical systems

#### 3.1.1 Measurement setup

Figure 3 shows a general setup for measuring the frequency response matrix (FRM) of a linear time-invariant (LTI) system operating in open (black) or closed (black and grey) loop. Since the reference signal \( r(t) \) is periodic, any deviation from the periodic behaviour is considered as being noise. The input-output DFT spectra of the measured steady state response are given by

\[
U(k) = U_0(k) + N_U(k), \quad Y(k) = G(\Omega_k)U_0(k) + N_Y(k)
\]  

where the input-output errors \( N_U(k) \) and \( N_Y(k) \) depend on all the noise sources in Figure 3. Note that \( N_U(k) \) and \( N_Y(k) \) are correlated via the generator \( n_g(t) \), controller \( n_c(t) \), and process noise sources \( n_p(t) \), and possibly via common disturbances picked up by the data acquisition channels (the input-output measurement errors \( m_u(t) \) and \( m_y(t) \) are then correlated). Modelling the input-output errors as filtered discrete-time or band-limited continuous-time white noise, the DFT spectra \( N_U(k) \), \( N_Y(k) \) are related to the input-output noise dynamics \( H_U(\Omega) \), \( H_Y(\Omega) \) and the unobserved (band-limited) white noise inputs \( E_U(k) \), \( E_Y(k) \) as [8, 21]

\[
N_U(k) = H_U(\Omega_k)E_U(k) + T_{H_u}(\Omega_k), \quad N_Y(k) = H_Y(\Omega_k)E_Y(k) + T_{H_y}(\Omega_k)
\]  

where \( T_{H_u}(\Omega), X = U, Y, \) represents the noise leakage (transient) error of the DFT. \( H_X(\Omega) \) and \( T_{H_X}(\Omega) \) are, respectively, rational matrix and vector functions of \( \Omega \) and, hence, are smooth functions of the frequency. \( T_{H_X}(\Omega) \) decreases to zero as an \( O(N^{-1/2}) \), while \( H_X(\Omega_k)E_X(k) \) is an \( O(N^0) \). These properties of noise model (12) are essential for the local polynomial method discussed in the sequel of Section 3.

#### 3.1.2 Single experiment

The local polynomial method starts from the input-output DFT spectra of \( NP \) samples (\( P \geq 2 \) periods of \( N \) samples) of the steady state response to a broadband periodic excitation signal

\[
Z(k) = \frac{1}{\sqrt{NP}} \sum_{t=0}^{PN-1} z(t) e^{-j\frac{2\pi k t}{PN}}
\]  

with \( z(t) \) a \((n_y + n_u) \times 1\) vector of the outputs and inputs stacked on top of each other

\[
z(t) = \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}
\]  

Since the noiseless DFT spectrum \( Z_0(k) \) is exactly zero at DFT frequencies \( kP + m \), for \( k = 0, 1, \ldots, N/2 - 1 \) and \( m = 1, 2, \ldots, P - 1 \), it can only contain signal energy at DFT frequencies \( kP \) (see Figure 4, left plot).
Therefore, $2n$ non-excited DFT lines $Z(kP + m)$ - the $n$ first lines to the left and the $n$ first lines to the right of $Z(kP)$ (see Figure 4, right plot, odd DFT frequencies) - are used for estimating the noise covariance matrices $C_Z(kP)$ and the noise transient terms $T_Z(\Omega_{kP})$

$$C_Z(kP) = \text{Cov} \left( \begin{bmatrix} H_Y(\Omega_{kP}) & E_Y(kP) \\ H_U(\Omega_{kP}) & E_U(kP) \end{bmatrix} \right) \quad \text{and} \quad T_Z(\Omega_{kP}) = \begin{bmatrix} T_Y(\Omega_{kP}) \\ T_U(\Omega_{kP}) \end{bmatrix}$$

in (12). This is done via a local polynomial least squares approximation of degree $R \geq 1$ of the transient terms

$$T_{Hz}(\Omega_{kP+m}) = T_{Hz}(\Omega_{kP}) + \sum_{r=1}^{R} t_r(k) m^r + \frac{1}{\sqrt{PN}} O(N_1^{-(R+1)})$$

where $N_1 = NP/m$. Removing the estimated noise transient terms from the signal lines $Z(kP)$ (see Figure 4, right plot, even DFT frequencies) finally gives the sample mean of the input-output DFT spectra. The whole procedure is a straightforward extension of the single-input, single-output case [1] and is summarised in the sequel of this section.

The noise transient $\hat{T}_Z(\Omega_{kP})$ and the noise covariance $\hat{C}_Z(kP)$ estimates are calculated from the $(n_y + n_u) \times (R+1)$ local polynomial least squares approximation $\hat{\Theta}$ of the noise transient as

$$\hat{\Theta} = Z_n K_n^H (K_n K_n^H)^{-1}, \quad \hat{T}_Z(\Omega_{kP}) = \hat{\Theta}_{[\cdot,1]}, \quad \hat{C}_Z(kP) = \frac{1}{q} (Z_n - \hat{\Theta} K_n) (Z_n - \hat{\Theta} K_n)^H$$

with $q = 2n - (R + 1)$, $x^H$ the complex conjugate transpose of $x$, and $x_{[\cdot,1]}$ the first column of $x$. The $(n_y + n_u) \times 2n$ matrix $Z_n$ and the $(R+1) \times 2n$ matrix $K_n$ have the following form

$$X_n = \begin{bmatrix} X(kP-m_n) & \cdots & X(kP-m_1) & X(kP+m_1) & \cdots & X(kP+m_n) \end{bmatrix}$$

where $Z(kP \pm m_i)$ are the input-output DFT spectra at the non-excited DFT lines; $m_i, i = 1, 2, \ldots, n$, are the first $n$ numbers of the set $\mathbb{N} \setminus \{kP \mid k \in \mathbb{N}\}$ (e.g. the grey arrows at the odd lines in Figure 4); and where $K(kP + m_i) = [1, m_i, \ldots, m_i^R]^T$, with $R$ the order of the polynomial approximation and $x^H$ the transpose of $x$. Note that the least squares estimate $\hat{\Theta}$ in (17) is calculated in a numerically stable way via the singular value decomposition of $K_n^H$. Combining (15) and (17) gives the following input-output noise covariance and transient estimates (for notational simplicity we remove the frequency argument)

$$\hat{C}_Y = \hat{C}_{Z[1,n_y,1,n_y]}, \quad \hat{C}_{YU} = \hat{C}_{Z[1,n_y,n_y+1,n_y+n_u]}, \quad \hat{C}_U = \hat{C}_{Z[n_y+1,n_y+n_u,n_y+1:n_y+n_u]}$$

$$\hat{T}_{Hz} = \hat{T}_{Z[1,n_y,1]}, \quad \hat{T}_{HU} = \hat{T}_{Z[n_y+1:n_y+n_u,1]}$$

where $X_{[m,p,q]}$ selects rows $n$ to $m$, and columns $p$ to $q$ of the matrix $X$. Subtracting (20) from $Z(kP)$ defines the sample mean of the input-output spectra over the $P$ periods

$$\hat{X}(kP) = X(kP) - \hat{T}_{Hz}(\Omega_{kP})$$

with $X \in \{Y, U\}$. The sample covariance of the sample mean (21) is related to the sample noise covariances (19) as
\[ \hat{C}_{XL}(kP) = \mu_{\text{poly}} \hat{C}_{XL}(kP) \quad \text{with} \quad \mu_{\text{poly}} = 1 + \| \Sigma_K^{-1} V_H^H K_{[1,:]} \|_2^2 \]  
(22)

with \( X, L \in \{ Y, U \}, \) \( C_{XX} = C_X, \) \( \| x \|_2^2 \) the 2-norm of \( x, \) \( U_k \Sigma_k V_k^H \) the singular value decomposition of \( K_k^H \), and where the factor \( \mu_{\text{poly}} \) quantifies the increase in noise variance of the sample mean \( \hat{X}(kP) \) w.r.t. the DFT spectrum \( X(kP) \) without transient suppression (proof: follow the same lines of [1], Appendix 8.2). The variance increase \( \mu_{\text{poly}} \) is typically 1 dB [1].

The expected value of the sample mean (21) and the sample noise covariance (19) equal

\[ E\{ \hat{X}(kP) \} = X_0(kP), \quad E\{ \hat{C}_{XL}(kP) \} = C_{XL}(kP) + O_{\text{leak}} \left( N_2^{-2(R+2)} \right) + O_{\text{int}} \left( N_2^{-2} \right) \]  
(23)

with \( N_2 = PN/m_n, \)

\[ C_{XL}(kP) = H_X(\Omega_{kP}) E\{ E_X(kP) E^*_H(kP) \} H^*_L(\Omega_{kP}) \]  
(24)

the true noise covariance matrix, and where \( O_{\text{leak}} \) and \( O_{\text{int}} \) are the bias contributions of, respectively, the residual noise leakage error and the noise interpolation error (proof: follow the same lines of [2], Appendix VI). The latter stems from the noise colouring over the \( 2n \) non-excited DFT lines in the linear least squares estimate (17).

Finally, the estimated input-output Fourier vector coefficients \( \hat{X}_k \) and their sample covariances \( \hat{C}_{\hat{X}_k L_k} \) are obtained by appropriate scaling of (21) and (22)

\[ \hat{X}_k = \frac{1}{\sqrt{PN}} \hat{X}(kP), \quad \hat{C}_{\hat{X}_k L_k} = \frac{1}{PN} \hat{C}_{XL}(kP) \]  
(25)

(the true Fourier coefficients are independent of the number of periods \( P \), which simplifies the comparison of experiments over different periods).

The degrees of freedom (dof\textsuperscript{noise}) of the sample noise covariance matrices (19), (22), and (25) are given by

\[ \text{dof}_{\text{noise}} = 2n - (R + 1) \]  
(26)

To ensure that \( \hat{C}_2(kP) \) in (17) has full rank one needs to fulfil \( \text{dof}_{\text{noise}} \geq n_x + n_u \). It follows that the frequency width \( [kP - m_u, kP + m_u] \) of the local polynomial least squares approximation is larger for multivariable than for single-input, single-output systems. However, \( P = 2 \) periods remain sufficient for the noise analysis.

### 3.1.3 Multiple experiments

Here we assume that experiments with \( n_u \) linearly independent broadband input signals \( u_e(t), e = 1, 2, \ldots, n_u \), are available (e.g. Hadamard or orthogonal multisines). The local polynomial method of Section 3.1.2 is then applied to each experiment separately, giving the following Fourier coefficients and their sample covariances

\[ \hat{X}_k^e, \quad \hat{C}_{\hat{X}_k L_k} \quad \text{for} \quad e = 1, 2, \ldots, n_u \]  
(27)

with \( X, L \in \{ Y, U \} \). The \( n_u \) values (27) are used separately in parametric transfer function modelling (see [17]); while the sample noise covariances \( \hat{C}_{\hat{X}_k L_k} \) (not the Fourier coefficients \( \hat{X}_k^e \)) are averaged over \( e \) for calculating the covariance of the nonparametric FRM estimate (4) (see Section 4).
3.2 Nonlinear dynamical systems

3.2.1 Measurement setup

The setup for measuring the best linear approximation (BLA) of a nonlinear PISPO system is the open loop setup of Figure 3 where the LTI system is replaced by the PISPO system, and where the actuator should be linear. Replacing the nonlinear PISPO system by the BLA of Figure 2, it can be seen that the input-output DFT spectra of the measured steady state response are given by

\[ U(k) = U_0(k) + N_U(k), \quad Y(k) = G(\Omega_k)U_0(k) + Y_s(k) + N_Y(k) \]  

(28)

where the input-output errors \( N_U(k) \) and \( N_Y(k) \) depend on the noise sources of the open loop setup in Figure 3 and satisfy (12). The zero mean stochastic nonlinear distortion \( Y_s(k) \) is uncorrelated with \( U_0(k) \), is not subject to transient (leakage) errors \( y_s(t) \) is a periodic function with the same period as \( r(t) \), and has a smooth (over the frequency) covariance matrix \( \text{Cov}(Y_s(k)) \) \cite{8, 12}. These properties of \( Y_s(k) \) are essential for the robust method discussed in Section 3.2.2.

3.2.2 Robust method

Here we assume that the \( n_u \) experiments are performed with the full random phase multisines (10), and that the reference signal \( r(t) \) is available. Since \( y_s(t) \) has the same periodicity as the reference signal \( r(t) \), no information about \( y_s(t) \) can be gathered by comparing consecutive periods of these experiments. Therefore, the experiments with the full random orthogonal multisines (10) are repeated for \( M \geq 2 \) independent random phase realisations \( \angle R_{\phi_p}(k) \) as well as \( \phi_e(k) \) in (10), for \( p, e = 1, 2, \ldots, n_u \) and \( k = 1, 2, \ldots, F \). Since \( y_s(t) \) depends on the particular random phase realisation of \( r(t) \), comparing the FRM estimates over the \( M \) different full random phase multisine experiments allows to estimate \( \text{Cov}(Y_s(k)) \). The whole procedure is explained in detail in the sequel of this section.

Applying the local polynomial method of Section 3.1.3 to each realisation of the full random phase multisine experiment gives the following sample means and sample noise covariances of the input-output DFT spectra

\[ \hat{X}^{[m]}(kP), \quad \hat{C}^{[m]}_{\text{noise}}(kP) \]

\[ \hat{X}^{[m]}(kP) = \left[ \hat{X}^{[m,1]}(kP), \quad \hat{X}^{[m,2]}(kP), \quad \ldots, \quad \hat{X}^{[m,n_u]}(kP) \right] \]

(29)

(30)

with \( X, L \in \{Y, U\} \), and for \( e = 1, 2, \ldots, n_u \), and \( m = 1, 2, \ldots, M \). Averaging of \( \hat{C}^{[m]}_{\text{noise}}(kP) \) over the realisations gives an improved estimate of the noise covariances

\[ \hat{C}^{[e, \text{noise}}_{XL}(kP) = \frac{1}{M} \sum_{m=1}^{M} \hat{C}^{[m]}_{XL}(kP) \]  

(31)

Straightforward averaging of the input-output DFT coefficients \( \hat{X}^{[m]}(kP) \) over the realisations \( m \) is not possible because of the random choice of the phases of the reference signal (10) over \( m \). To allow for averaging over \( m \), the input-output spectra \( \hat{X}^{[m]}(kP) \) must first be referred to the reference signal. This is done as follows. First, note that for each random realisation \( m \), the full random orthogonal multisines (10) can be written under the form

\[ R^{[m]}(k) = D_{\text{Ampl}}(k)T^{[m]}_{\text{Phase}}(k) \]  

(32)
where $D_{\text{Ampl}}(k)$ is a diagonal matrix containing the - by construction - realisation independent input amplitude spectra ($D_{\text{Ampl}[p,q]}(k) = |R_{[p,q]}(k)|^2 \delta_{pq}$), and where $T_{\text{Phase}}^{[m]}(k)$ is an orthogonal matrix (\((T_{\text{Phase}}^{[m]})^{-1} = T_{\text{Phase}}^{[m]H}(k)\)) containing the realisation dependent phases of the inputs. It is related to the phases of (10) as

$$T_{\text{Phase}}^{[m]}(k) = D_{\text{Ampl}}^{[m]}(k) T D_{\phi}^{[m]}(k)$$

(33)

with $T$ defined in (7), and $D_{\in \mathbb{R}[p,q]}^{[m]}(k) = e^{j\angle R_{[p,q]}^{[m]}(k) \delta_{pq}}$, for $p, q = 1, 2, \ldots, n_a$ and $m = 1, 2, \ldots, M$. Next, the input-output DFT coefficients $\hat{X}^{[m]}(kP)$ are left divided by $T_{\text{Phase}}^{[m]}(k)$ giving

$$\hat{X}^{[m]}_R(kP) = \hat{X}^{[m]}(kP) T_{\text{Phase}}^{[m]H}(k)$$

(34)

Note that vec($\hat{X}^{[m]}_R(kP)$) and vec($\hat{X}^{[m]}(kP)$), where vec($X$) puts the columns of $X$ on top of each other, have the same covariance matrix (proof: see [17]). Finally, the sample mean of the DFT spectra is obtained by averaging (34) over the realisations, and the total sample covariance of each column of $\hat{X}^{[m]}_R(kP)$ is calculated by an averaging over the $M$ realisations and $2n_R + 1$ neighbouring excited frequencies. The latter is necessary for getting a covariance estimate with sufficient degrees of freedom. This results in the following algorithm

$$C^{[e]}_{X_R}(kP) = \frac{1}{M} \sum_{m=1}^{M} \sum_{i=-n_R}^{n_R} r^{[m,e]}_X(k+k_i) t^{[m,e]H}_{L}(k+k_i)$$

(35)

$$r^{[m,e]}_X(k) = \hat{X}^{[m]}_R(kP) - \hat{X}^{[m]}_{R\ldots[e]}(kP)$$

$$\hat{X}^{[m]}_R(kP) = \frac{1}{M} \sum_{m=1}^{M} \hat{X}^{[m]}_R(kP)$$

(36)

where $X_{[\ldots,e]}$ denotes column $e$ of $X$, and with $k_i, i = -n_R, \ldots, -1$, the first $n_R$ excited harmonics left from $k$, $k_0 = 0$, and $k_i, i = 1, \ldots, n_R$, the first $n_R$ excited harmonics right from $k$.

Taking the expected value of (31), (35), and (36), and neglecting the non-dominant bias error terms in the covariances, we find

$$E \left\{ \hat{X}_R(kP) \right\} = X_0(kP)$$

(37)

$$E \left\{ C^{[e]}_{X_R}(kP) \right\} = \mu_{\text{poly}} C_{X_L}(kP) + P C_{X_L}(kP) + O_{\text{int}} (N^\alpha_3)$$

(38)

$$E \left\{ C^{[e], \text{noise}}_{X_L}(kP) \right\} = C_{X_L}(kP) + O_{\text{int}} (N^\alpha_2)$$

(39)

with $N_3 = N / n_t$, $\alpha = 1, 2$ for respectively non-uniformly and uniformly distributed excited harmonics, $N_2 = PN / n_t$, $C_{X_L}(kP)$ the true noise covariance matrix (24), $C_{X_L}(kP)$ the true covariance matrix of the stochastic nonlinear distortions, and with $\mu_{\text{poly}}$ defined in (22) (proof: follow the same lines of [1], Appendix 8.3). $O_{\text{int}} (N^\alpha_3)$ represents the bias contribution originating from the averaging of the total covariance matrix over the $2n_R+1$ neighbouring excited frequencies $k+k_i, i = -n_R, \ldots, n_R$.

Appropriate scaling of the input-output DFT spectra (36) and their covariance matrices (31) and (35) finally gives the local polynomial estimate of the input-output Fourier vector coefficients and their sample covariances.
for \( e = 1, 2, \ldots, n_u \), and \( X, L \in \{ Y, U \} \), where (41) and (42) are, respectively, the total and the noise sample covariance matrices of \( \hat{X}_{k}, \hat{L}_{k} \). Comparing (38) with (39), it can easily be seen that an estimate \( \hat{C}_{X_kL_k} \) of the covariance matrix of the stochastic nonlinear distortions w.r.t. one multisine experiment is found by subtracting the noise covariance (42) from the total covariance (41), and multiplying the difference by the number of realisations \( M \).

\[
\hat{C}_{X_kL_k} = M \frac{1}{n_u} \sum_{e=1}^{n_u} \left( \hat{C}_{X_kL_k} - \hat{C}_{X_kL_k}^{\text{noise}} \right)
\]

(43)

To reduce the variability of the estimate, the differences are averaged over the \( n_u \) experiments.

The degrees of freedom of the noise (31), (42) and total (35), (41) sample covariance matrices are given by, respectively,

\[
\text{dof}^{\text{noise}} = M (2nR + 1), \quad \text{dof} = (M - 1) (2nR + 1)
\]

(44)

Since the variance of the sample covariance estimates decreases as \( 1/\text{dof} \), one could make the degrees of freedom as large as possible by an appropriate choice of \( n, R, \) and \( n_R \). The drawback of this strategy is the increase of the bias interpolation errors in (38) and (39) for increasing values of \( n \) and \( n_R \). Although the required minimal values of dofnoise and dof are dictated by the system identification application (see [17]), \( M = 2 \) realisations and \( P = 2 \) periods remain sufficient for estimating accurately the level of the noise and the stochastic nonlinear distortions on the Fourier coefficients. The main difference with the single-input, single-output case is that larger values of \( n \) and \( n_R \) are needed to achieve a given accuracy on the covariance estimates (e.g. the full rank condition of the covariance matrix (17) requires that \( \text{dof}^{\text{noise}} \geq n_u + n_y \)).

Note that the estimated Fourier coefficients and their sample covariances are correlated over the frequency. The correlation length is \( \pm 2n/R \) for the sample mean (40) and the sample noise covariance matrix (42), while it is \( \pm 2nR \) for the total sample covariance (41).

### 3.3 Special case: the generator errors are dominant

Since the contribution of the generator errors \( n_g(t) \) to the input-output errors \( H_U (\Omega_k) E_U (k) \) and \( H_Y (\Omega_k) E_Y (k) \) in (12) are related by the true system transfer function (see Figure 3), they do not contribute to the covariance of the FRM measurement (proof: see [17]). If the generator errors are dominant, then the covariance of the FRM measurement is small because the large generator error contributions to the input-output covariance matrices are cancelled in the covariance expression of the FRM (see [17]). Hence, the slightest error made in the input-output covariance estimates causes a large error in the covariance estimate of the FRM (the difference of large terms should almost be zero). This high sensitivity to small errors on the input-output covariance estimates is avoided by performing the sample mean and sample covariance calculations on the FRM instead of on the input-output DFT spectra. The bias introduced by the division (4) in the modified procedure can be neglected if the input signal-to-noise ratio is larger than 10 dB [8].

In modal analysis experiments, the nonlinear distortions generated by the shakers act as dominant generator errors that contribute to the total input-output covariances but not to the noise input-output covariances (see
Section 2.2: the nonlinear distortions are periodic signals that are uncorrelated with the reference signal \( r(t) \). Since the robust method of Section 3.2.2 introduces small interpolation errors in the total covariance estimates (see (38)), \( \hat{X}_K^{[m]} \) and \( \hat{X}_{[\epsilon]}^{[m]} \) in the sample mean and sample covariance calculations (35) and (36) are replaced by respectively \( G^{[m]}(\Omega_k) \) and \( \text{vec}(G^{[m]}(\Omega_k)) \), where \( \text{vec}(X) \) puts the columns of the matrix \( X \) on top of each other, and with \( G^{[m]}(\Omega_k) \) the FRM of the \( m \)th experiment with the full random orthogonal multisines (10). Although the input-output noise covariances are also subject to small interpolation errors (see (39)), they are still calculated via the procedure of Sections 3.1.2 and 3.1.3, because the generator noise is mostly not dominant in modal analysis experiments.

### 3.4 Non-steady state conditions

If the system is measured under transient conditions, then (28) is replaced by

\[
U(k) = U_0(k) + N_U(k) + T_{G_U}(\Omega_k), \quad Y(k) = G(\Omega_k)U_0(k) + Y_s(k) + N_Y(k) + T_{G_Y}(\Omega_k)
\]

where \( T_{G_U}(\Omega_k) \) and \( T_{G_Y}(\Omega_k) \) are, respectively, \( n_u \times 1 \) and \( n_y \times 1 \) rational vector functions depending on the actuator and system dynamics, and on the difference between the initial and final conditions of the experiment [8]. Replacing \( N_U(k) \) and \( N_Y(k) \) in (45) by (12), it can be seen that no distinction can be made between the noise and the system transient (leakage) errors. Hence, the noise analysis of Section 3 remains valid and can be applied to the first two periods of the transient response to periodic inputs. For lowly damped systems this results in a significant reduction of the measurement time.

### 4 Nonparametric FRF modelling

For experiments performed with the orthogonal and full random phase multisines, the covariance matrix of the measured FRM (4) is calculated as

\[
\text{Cov}(\text{vec}(\hat{G}(\Omega_k))) \approx \left( \hat{U}_k \hat{U}_k^H \right)^{-1} \otimes \left( V_k \hat{C}_L V_k^H \right), \text{ with } V_k = \left[ I_{n_y} \quad -\hat{G}(\Omega_k) \right]
\]

with \( \hat{U}_k \) the input Fourier vector coefficients (25) or (40), and \( \hat{C}_L^{[\epsilon]} \), \( X, L = \{ Y, U \} \), the corresponding noise (25) or total (41) sample covariances (proof: see [17]). Evaluating (46) with (43) quantifies the level of the stochastic nonlinear distortions on the FRM measurement w.r.t. one multisine experiment. Note that the degrees of freedom of the covariance estimate (46) equal those of \( \hat{C}_L^{[\epsilon]} \) (44) multiplied by \( n_u \).

It can easily be verified that the FRM estimate (4) is unbiased in the absence of input measurement noise and plant transient (leakage) errors. If the input signal-to-noise ratio is at least 10 dB, then the relative bias error is smaller than \( 5 \times 10^{-5} \) [8] and, hence, can be neglected. In the presence of plant transient (leakage) errors, the expected value of the FRM estimate (4) is given by

\[
E \left\{ \hat{G}(\Omega_k) \right\} = G(\Omega_k) + O_{\text{leakG}} \left( \sqrt{N N_2}^{-[R+2]} \right)
\]

with \( N_2 = PN/m_n \) and where \( O_{\text{leakG}} \) is the bias contribution of the plant leakage errors (proof: see [17]). Note that the noise transient (leakage) errors do not introduce a bias in the FRM estimates.
5 Measurement example

The goal of the measurement example is to illustrate the importance of the system and/or noise transient (leakage) errors in nonparametric FRM measurement and parametric transfer function modelling of lightly damped systems. Therefore, an aluminium tooling plate (PE 200) of size 30.4 cm x 61.8 cm x 6.7 mm is excited under free-free boundary conditions by 2 mini-shakers (B&K 4810) spaced at 25.5 cm apart (see Figure 5, left picture). The mini-shakers are connected to the aluminium plate via an impedance head (B&K 8001) and a plexiglass stinger rod of 1.8 cm length (the visible part is 1 cm long) glued into screws of 1 cm length with a bore hole of 4 mm deep (see Figure 5, right picture). To limit the inductive loading of the arbitrary waveform generators (HP E1445A, $Z_{\text{out}} = 50\,\Omega$), a $20\,\Omega/5\,\text{W}$ resistor is put in series of each mini-shaker. The force and acceleration signals measured by the impedance heads are amplified (B&K 2635 charge amplifiers) and buffered ($Z_{\text{in}} > 5\,\text{M}\,\Omega$, $Z_{\text{out}} = 50\,\Omega$) before being applied to the data acquisition channels (HP E1430A, $Z_{\text{in}} = 50\,\Omega$). The generator and acquisition units are synchronised and operate at the sampling frequency $f_s = 10\,\text{MHz}/2^{12} \approx 2.44\,\text{kHz}$.

The robust method of Section 3.3, with $P = 2$ periods and $M = 2$ independent realisations, is applied to this $n_u = 2$ input, $n_y = 2$ output system. The full random phase multisines (10) have a flat amplitude spectrum ($D_{\text{Ampl}}(k)$ in (32) is independent of $k$) with equal rms values for all inputs ($r_1\text{rms} = r_2\text{rms} = 1.2\,\text{V}$), and contain $F = 25772$ harmonically related frequencies that are uniformly distributed in the band $[120\,\text{Hz}, 600\,\text{Hz}]$ ($k f_s/N$ with $k = 6442, 6443, \ldots, 32213$ and $N = 128 \times 1024$ points per period). The frequency resolution of the corresponding FRM measurement is $18.6\,\text{mHz}$. For each experiment the force (= input) and acceleration (= output) signals are measured after a waiting time of $N/64$ samples.

The force-to-acceleration FRM is calculated using the procedure of Section 3.3, with $R = 2$, and $n, n_R$ in (44) chosen such that $\text{dof}^{\text{noise}} = \text{dof} = 10$. Figure 6 shows the results. It can be seen that $G_{[1,1]}$ contains less resonances than the other entries of the FRM. Note also the presence of the third mains harmonic (150 Hz) in the noise variance of $G_{[2,1]}$ and $G_{[2,2]}$. The reason for this is the required high gain of the charge amplifier corresponding to the second acceleration signal.

To estimate the importance of the system and/or noise transient (leakage) in the data, the estimates are repeated using the procedure of Section 3.3 without removing the transient in the sample mean (21) and sample noise covariance estimates (17): $X(kP) = X(kP)$ and $\hat{C}_Z(kP) = \frac{1}{2n+1}Z_n^H Z_n^T$, where $Z_n$ is defined in (18). Figure 7 shows the ratio of the variance of the FRM without transient suppression to the variance of the FRM with transient suppression. At the resonances the uncertainty of the estimates without transient suppression is about 30 dB (20 dB) larger for the noise (noise + nonlinear distortion) errors. In those frequency bands and FRM entries where the transient (leakage) errors can be neglected, the noise uncertainty of the FRM estimates with transient suppression is about 1 dB larger than that without transient suppression. This observation is consistent with the increase in uncertainty quantified by $\mu_{\text{poly}}$ in (22). Figure 8 shows a zoom around the first resonance peak of the FRM estimates with (top rows) and without (bottom rows) transient
Figure 6: FRM measurement with transient suppression (black), its noise variance (light grey), and its total variance (dark grey).

Figure 7: Left figures: ratio of the noise (light grey) and total (dark grey) variances of the FRM measurement without and with transient suppression. Right figures: ratio of the total variance of the FRM measurement (46) calculated via the input-output total sample covariances (35), to that calculated via the procedure of Section 3.3.
suppression. It turns out that the peak consists of 2 very closely spaced resonance frequencies. While the nonlinear distortions are clearly visible in the estimates with transient suppression (the total variance is about 8 to 15 dB larger than the noise variance in the top rows), this is no longer the case in the estimates without transient suppression (see the bottom rows: the light and dark grey lines coincide everywhere). It nicely illustrates the importance of system and/or noise leakage elimination in lowly damped systems; even for long data records.

To illustrate the impact of the nonlinear distortions generated by the shakers on the total covariance estimate of the FRM, the total covariance calculated via the procedure of Section 3.3 is compared in Figure 8 with the total covariance estimate (46) obtained via the input-output total sample covariances (35). It can be seen that at the resonance frequencies, the total variance predicted via the input-output total sample covariances (35) is 10 to 15 dB too large. Hence, at those frequencies, the nonlinear distortions of the generator are the dominant error source.

In each of the following frequency bands [247 Hz, 254 Hz], [378 Hz, 388 Hz], [430 Hz, 440 Hz], [450 Hz, 455 Hz], and [490 Hz, 520 Hz], a 2 input, 2 output continuous-time common denominator transfer function

$$G(s, \theta) = \sum_{m=0}^{n_b} B_m s^m \sum_{m=0}^{n_a} A_m s^m$$

(49)
of order \(n_b/n_a = 8/6, 6/4, 4/2, 4/2,\) and \(10/8\) respectively, is identified by minimising the sample maximum likelihood cost function w.r.t. \(\theta\) (see [17]). This is done for the frequency domain data (sample means and total sample covariances of the FRM) with and without nonparametric transient (leakage) removal. Table 1 gives the estimated resonance frequencies and damping ratios. It can be seen that the estimates of the resonance frequencies with and without nonparametric transient elimination do not coincide \((|\Delta \hat{f}_0| > 2\text{std}(\hat{f}_0))\), with \(\text{std}(\hat{f}_0)\) the standard deviation of the estimates without transient elimination). This is also the case for the damping ratio of the second resonances \((|\Delta \hat{\zeta}| > 2\text{std}(\hat{\zeta})\) with \(\text{std}(\hat{\zeta})\) the standard deviation of the estimates without transient elimination), while those of resonances 1, 3, 4, and to 5 do coincide \((|\Delta \hat{\zeta}| \leq 2\text{std}(\hat{\zeta}))\). Note also that the uncertainty of the estimates without transient suppression is about 1.9 to 8.6 times larger than that of the estimates with transient elimination (see “ratio std” in Table 1). Both the bias and the increased variability of the estimates without transient elimination are due to the system and/or noise transients in the measured input-output data.
Table 1: Estimated resonance frequencies $f_0$ and damping ratios $\zeta$ and their corresponding standard deviation. $\Delta$ is the difference between the estimate without and with nonparametric transient suppression, and “ratio std” is the ratio of the standard deviations of the estimates without and with nonparametric transient suppression.

6 Conclusion

We have shown that for lightly damped systems operating under steady state conditions, the noise transient (leakage) errors remain important - even for long data records - and should be suppressed in the input-output data. Two periods are sufficient to eliminate the system and/or noise transient (leakage) errors in the FRM via a local polynomial approximation of the input-output transient contributions. To characterise the noise level and the level of the nonlinear distortions in FRF measurements of multivariable ($n_u$ inputs, $n_y$ outputs) nonlinear systems, only 2 periods of the transient response to 2 independent realisations of full random orthogonal multisines are needed. As such, for a given experimental time, the loss in frequency resolution of the proposed measurement procedure w.r.t. the random excitation case [2, 3] is $4n_u$. This is a significant decrease w.r.t. the factor of at least $2(n_y+5)n_u$ for the classical method [8, 16]. Contrary to the random input case, no local polynomial approximation of the FRM is needed.

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