Simple methods and insights to deal with nonlinear distortions in FRF-measurements.

J. Schoukens*, J.Swevers**, J. De Cuyper***, Y. Rolain*

*VUB Department ELEC, Pleinlaan 2, 1050 Brussels, Belgium, johan.schoukens@vub.ac.be

** KUL, Department of Mechanical Engineering, Division PMA

***LMS International

Abstract - In this paper simple methods are described to detect, qualify and quantify the presence of nonlinear distortions. Their impact on FRF measurements is discussed The techniques are illustrated on experimental results.

I. Introduction

The goal of this paper is to provide the reader insight in the behaviour of nonlinear distortions and their impact on FRF measurements. This allows not only a better understanding of the error mechanism, the knowledge can also be applied to the experiment design in order to get the best results under the imposed operational conditions. To do so, the user should clearly specify the goal of his measurements. In order to formalize this discussion, we use the general structure given in Figure 1. The measured output y(t) consists of

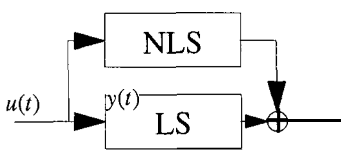

Figure 1:Principal setup.

a linear y_{L} and a nonlinear y_{NL} contribution. For simplicity we assume that

$$\lim_{(u)_{\mathrm{RMS}} \to 0} \frac{(y_{\mathrm{NL}})_{\mathrm{RMS}}}{(y_{\mathrm{L}})_{\mathrm{RMS}}} = 0\,, \tag{1}$$

so that the linear contribution dominates over the nonlinear one for sufficiently small inputs. Under this assumption we have two basic options: 1) the goal of the measurement is to get the FRF of the underlying linear system, minimizing the impact of the NLS on the measurements; 2) try to find the best linear approximation to the global system, including the NLS. The first option is the best choice if some underlying linear physical model exists and the user wants to identify it as good as possible. The second choice is preferred if the model will be used to describe the relation between input and output using linear models. At that moment, the nonlinearity can be linearised around the operation point of the test. The best linear approximation is called the related dynamic system (RLDS). It will be defined more formally in the next Section.

This work was supported by the FWO-Vlaanderen, the Flemish community (Concerted action IMMI), and the Belgian government (IUAP-4/2).

II. Mathematical framework

A. A model for the nonlinear distortions

In this Section we provide a brief description of the mathematical framework used to describe the nonlinear distortions. A detailed description, stating precisely all underlying assumptions is given in [9]. The nonlinear distortions are described by a Volterra series. An extended introduction to this technique is given in the book of Schetzen (1980). The basic idea is to describe the output using multi-dimensional convolutions, e.g., up to the third degree

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} h_1(\tau)u(t-\tau)d\tau \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_2(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)d\tau_1 d\tau_2 \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_3(\tau_1,\tau_2,\tau_3)u(t-\tau_1)\dots u(t-\tau_3)d\tau_1 d\tau_2 d\tau_3 \end{aligned} \tag{2}$$

For periodic excitations with N harmonics at frequencies kf_{max}/N, $k = 1, \dots, N$, relation (2) simplifies to a sum over all possible frequency combinations adding to the output Fourier coefficient Y(l) at frequency lf_{max}/N [3]: $Y(l) = \sum_{\alpha=1}^{\infty} Y^{\alpha}(l)$, with $Y^{\alpha}(l)$ the contribution of degree a ,

$$Y^{\alpha}(l) = \sum_{k_1=-N}^{N} \dots \sum_{k_{\alpha-1}=-N}^{N} H^{\alpha}_{L_k, k_1, \dots, k_{\alpha-1}} X(k_1)\dots X(k_{\alpha-1})X(L_k)$$

and $L_k = l - \sum_{i=1}^{\alpha-1} k_i$. (3)

H^{α} is the symmetrized frequency domain representation of the Volterra kernel of degree α [7] so that the order of the frequencies $L_k, k_1, k_2, \ldots, k_{\alpha-1}$ has no importance. For simplicity, we consider only the set systems S for which the infinite sums in the Volterra series converge for the considered inputs.

B. Class of excitation signals

We study the behaviour of the NLS for random multisine excitations. This is a periodic random excitation with a user defined choice for the amplitude spectrum. All the results can be easily generalized to periodic random signals (random amplitude and random phase), at a price of adding an additional expectation with respect to the amplitudes to the expressions as is shortly commented in [9]. This generalizes the results to the wider class of normally distributed random excitations. It can be even shown that under some conditions (e.g. Wiener systems with non zero dynamics) the amplitude distribution of the noise does not strongly influence the results, so that the RLDS obtained with random multisines is also valid for other excitations like uniform or binary distributed noise.

Definition 2.2: A signal x(t) is a normalized random multisine belonging to E_N if

$$x(t) = \sum_{k=-N}^{N} X(k) e^{j2\pi \frac{f_{max}}{N} kt}, \quad (4)$$

with $x(k) = x(-k) = |X(k)| e^{j\varphi_k}$, f_{max} the maximum frequency of the excitation signal, N the number of frequency components, and the phases φ_k a realization of an independent uniformly distributed random process on $[2\pi[$.

Since we study also the asymptotic behaviour for multisines with a growing number of harmonics ($N \to \infty$), the signals are scaled with $1/\sqrt{N}$: $|X(k)| = \hat{X}(kf_{max}/N)/\sqrt{N}$, where $\hat{X}(f) \in \mathbb{R}^+$ is a uniformly bounded function. For simplicity X(0) is set to zero, considering the DC component as the operating point of the system.

III. Asymptotic properties of non-parametric FRF

Again this section contains a very brief summary of the results obtained in [9]. The FRF measurement G(2) at frequency f_l for nonlinear systems is the sum of the nonlinear contributions of degree a, $G^{\alpha}(l)$:

$$G(l) = \sum_{\alpha=1}^{\infty} G^{\alpha}(l), \text{ with } G^{\alpha}(l) = Y_l^{\alpha}/X_l \text{ and}$$

$$G^{\alpha}(l) = \sum_{k_1=-N}^{N} \ldots \sum_{k_{\alpha-1}=-N}^{N} H^{\alpha}_{L_k, k_1, k_2, \ldots, k_{\alpha-1}} \frac{X(k_1)\ldots X(k_{\alpha-1})X(L_k)}{X(l)} \quad (5)$$

A. Best linear approximation (RLDS) and nonlinear noise contributions (SNLS)

In [9] it is shown that the nonlinear distortions split in two classes: systematic and stochastic contributions.

- systematic contributions: there exists a related linear dynamic system (RLDS) to which the expected value of the FRF estimate converges under weak conditions. This RLDS is also linked to the classical results where the system is excited with normally distributed noise. It differs from the underlying linear system by the systematic contributions of the nonlinear distortions.
- stochastic contributions: even for a very large number of frequencies, the FRF estimate is not smooth as a function of the frequency. It is scattered around its expected value, and these deviations do not converge to zero. They are called the stochastic nonlinear distortions (SNLS).

So the measured FRF can be written as the sum of 3 parts:

$$G(\omega_k) = G_R(\omega_k) + G_S(\omega_k) + N_G(\omega_k), \quad (6)$$

with $G_R(\omega_k)$ the related dynamic system (RLDS), $G_S(\omega_k)$ the stochastic nonlinear contributions (SNLS), and $N_G(\omega_k)$ the errors due to the output noise. $G_R(\omega_k)$ consists on its turn of two parts:

$$G_R(\omega_k) = G_0(\omega_k) + G_B(\omega_k), \quad (7)$$

with $G_0(\omega_k)$ the underlying linear system and $G_B(\omega_k)$ the bias or systematic errors due to the nonlinear distortions. $G_S(\omega_k)$ is called a stochastic contribution since it behaves as uncorrelated (over the frequencies) noise although the reader should be aware that it is not really a noise component. Due to this noisy behaviour, the presence of nonlinear distortions is often not recognized. These different contributions to the FRF are studied in more detail below for two situations. In the first case we average over different realizations of the random multisine, keeping the frequency grid and the amplitude spectrum X(f) of the excitation signal constant (Theorem 1). The second case deals with the asymptotic behaviour if the number of harmonics $N \to \infty$ (Theorem 2).

B. Averaging over different realizations of the excitation

Theorem 1: For a system belonging to the system set S, excited with independent realizations of a normalized random multisine $x_N \in E_N$, the expected value of $G^{\alpha}(l)$

is given by:

$$E[G^{\alpha}(l)] = 2^{\frac{a-1}{2}} \sum_{s_1=1}^{N} \sum_{s_{\frac{\alpha-1}{2}}=1}^{N} H^{\alpha}_{l,-s_1,s_1,\dots,s_{\frac{\alpha-1}{2}}} |X(s_1)|^2 \dots \left|X(s_{\frac{\alpha-1}{2}})\right|^2 \tag{8}$$

if a is odd, and $E[G^{\alpha}(l)] = 0$ if a is even.
Proof: see [9].
Remark: as mentioned before, these results can be generalized to random amplitudes by considering and additional expected value over the amplitude spectra:

$$E[G^{\alpha}_l] =$$

$$2^{\frac{a-1}{2}} \sum_{s_1=1}^{N} \dots \sum_{s_{\frac{\alpha-1}{2}}=1}^{N} H^{\alpha}_{l,-s_1,s_1,\dots,s_{\frac{\alpha-1}{2}}} \mathcal{E}_{\text{amp}}\{X(s_1)^2 \dots X(s_{\frac{\alpha-1}{2}})^2\}, \tag{9}$$

where $\mathcal{E}_{\text{amp}}\{.\}$ denotes the expected value with respect to the amplitudes of the random multisines. The advantage of using deterministic amplitudes is that an additional averaging process is avoided.
The related dynamic system is then given by

$$G_R(\omega_l) = G_0(\omega_l) + \sum_{a=2} E[G^{2\alpha-1}(\omega_l)]\,. \tag{10}$$

This expression gives the relation between the best linear approximation, the nonlinear distortion and the power spectrum of the excitation.
In case of a Wiener-Hammerstein system [2], consisting of a linear system with transfer function $R(o)$, followed by a static non linearity (v(t) = $\sum_{k=0}^{\infty} a_k u^k(t)$, with $a_k \in \mathbb{R}$) and a second linear system S(o), the previous expressions can be further simplified ($N \to \infty$) [4] to

$$G_R(\omega_l) = \sum_{\alpha=1}^{\infty} G^{2\alpha+1}_{Bl} = R(l)S(l)C(X, R)\,, \tag{11}$$

with C(X, R) a frequency independent constant depending on the input power spectrum and R .

C. Asymptotic behaviour of the FRF if the number of harmonics $N \to \infty$

Instead of considering $G^{\alpha}_B(\omega_l)$ as the expected value (see previous section) it also can be interpreted as that part of the transfer function contribution of degree a that is independent of the random phase of the excitation. Consequently it is a deterministic component that models the systematic contribution of the nonlinear distortion to the FRF. $G^{\alpha}_S(\omega_l)$ depends on the random phase of the excitation, and consequently it is a random component modelling the stochastic contribution of the nonlinear distortion of degree a to the FRF. Neither of both contributions is decreasing if the number of frequencies N increases. This means that the FRF does not become smooth for $N \to \infty$.
Theorem 2: for a system, excited with a random multisine $x_N \in E_N$, $G^{\alpha}(\omega_l) = G^{\alpha}_B(\omega_l) + G^{\alpha}_S(\omega_l)$, where $G^{\alpha}_B(\omega_l)$ is an $O(N^0)$ and $G^{\alpha}_S(\omega_l)$ a $O_{MS}(N^0)$.
Proof: see [9].
The stochastic behaviour of $G^{\alpha}_S(\omega_l)$ can be further characterized, showing that its second, third and fourth order properties are completely similarly to those of the noise. This explains why it is difficult to distinguish between noise and nonlinear distortions.
Theorem 3: For a system excited with a random multisine $x_N \in E_N$ the following properties are valid:

For $k, l \neq 0$

i) $E[G_S(\omega_l)] = 0$

ii) $E[G_S(\omega_l)G_S(\omega_k)] = O(N^{-1})$ if $k \neq l$ and

$$E[|G_S(\omega_l)|^2] \equiv \sigma^2_{GS}(l) = O(N^0)$$

iii) $E[G_S(\omega_l)|G_S(\omega_k)|^2] = O(N^{-1})$

iv) $$E[(|G_S(\omega_k)|^2 - \sigma^2_{GS}(k))(|G_S(\omega_l)|^2 - \sigma^2_{GS}(l))] = \begin{cases} O(N^{-1}) & k \neq l \\ O(N^0) & k = 1 \end{cases}$$

Proof: see [9].
Remark: these observations are in agreement with the classical result showing that the output of a nonlinear system can be split in two parts ([1],[6]): a first part that is linearly related with the input (in our case leading to G_R), and a second part that is uncorrelated with the input (leading to G_S). Theorem 3 tells more about the second order properties of the uncorrelated part.

Conclusion: Measurements of the best linear approximation (G_R) are obtained by eliminating the stochastic nonlinear contributions (G,) and the noise contributions (N_G). Both stochastic contributions G_S and N_G can be reduced through averaging. By making them initially as small as possible, the required number of averages (and hence the measurement time) can be strongly reduced. This is studied in Section V.

IV. Experimental Setup

Based on the previous explained theoretic results, a series of simple tests and rules are provided to deal with nonlinear distortions. These are illustrated on a simulation or experiments. The experimental test setup consists of a rectangular plate to which two masses are attached. These masses are chosen such that the main resonance frequencies of the system lie in the frequency range between 10 Hz and 70 Hz. The plate is supported by means of leaf springs attached to each corner of the plate. One of the supports is an electro-dynamical shaker which provides excitation in the vertical direction. The input to the system is the voltage sent to the power amplifier of the shaker, which converts this signal linearly to a shaker current (proportional to the shaker force). The response of the test setup is measured by means of an accelerometer placed near to the point of excitation. The output voltages of one of these accelerometers, proportional with the measured acceleration, is used as output of the system.

V. A simple test to detect, qualify and quantify nonlinear distortions

The aim of this section is to provide a simple test to check the nonlinear behaviour of the device under test. We assume that the user is the master of the excitation signal which means that the interaction between the generator and the (nonlinear) device is small. This is for example typically the case in control applications where the excitation signal is a sequence in the memory of a computer. In other situations, where the excitation is for example the measured force, this assumption might be invalid. The nonlinear interaction can create additional undesired harmonic components in the excitation signal. At that moment it is necessary to compensate for these distortions (using for example a software feedback mechanism [10]), or to make a first order correction of the measurements. For brevity we do not discuss these techniques here.

The basic idea of this test is to excite the system with an odd-odd multisine, where only the frequencies $4k+1$, $k = 0, 1, 2, \ldots k_{max}$ in eq. (4) have amplitudes different from zero. Next the output spectrum is calculated with a DFT (implemented as an FFT) using a **rectangular** window. From (3) it follows that the even nonlinearities excite only the even harmonics at the output ($2k, k = 1, 2, \ldots$), while the odd nonlinearities appear only at the odd harmonics ($2k+1, k = 1, 2, \ldots$). Due to the choice of the excitation signal we get the following possibilities:

- at lines $4k+1$: the output consist of the linear contribution + odd nonlinear distortions
- at lines $4k+2$: only the even nonlinear distortions appear
- at lines $4k+3$: only odd nonlinear distortions appear

This allows already to get an idea of the nonlinear behaviour of the system. If at least $M \geq 2$ successive periods are measured in one block, it is still possible to make the same conclusions (respectively at lines $M(4k+1)$, $M(4k+2)$ and $M(4k+3)$). On top of that also the noise level can be characterized by looking at the lines that are NO multiple of M since these can't be excited by a signal with M periods in the window. So only the noise (having a non-periodic behaviour) can contribute there.

This experiment was done on the setup described in the previous section. The system was excited at the frequencies $f_k = (4k+1)f_0$, $k = 4, \ldots 35$ withconstant amplitude and random phase. $M = 6$ and 7 different realization were averaged (each time using another random generation for the phase). The results are shown in Fig. 2. This simple test learns a lot. There are significant

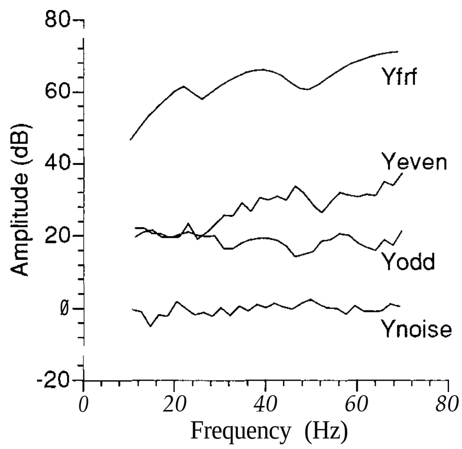

Figure 2:Detection of nonlinear distortions at the output. Yfrf: linear + odd nonlinear contr., Yeven: even nonlinear contr., Yodd: odd nonlinear contr., Ynoise: noise level

even and odd nonlinear distortions present. These are well above the noise level (that can be seen to be about 60 dB below the signal level). The even nonlinear distortions dominate. Since we know that they don't contribute to the best linear approximation, it is better to eliminate them in the later tests using odd multisines.

VI. Measuring the related dynamic system

In the next step the transfer function of the system is measured. In order to get a better insight on the excitation dependency of the error mechanism we first present some simulation results. Next we analyse the experimental results.

A. Simulation results

The FRF of a static nonlinear system $y = u + u^2/2.8 + u^3/15$ is measured using three different excitation signals with a flat power spectrum: a random noise (zero mean normally distributed), an odd (50 frequencies) and a consecutive (100 frequencies) multisine excitation all with an RMS value of 1. The power spectrum of these signals was band-limited with $f_{max} = 0.1 f_s$. Also a binary noise excitation was added by taking the sign of normally distributed noise. For all these signals the mean distortion $E = \frac{1}{N}\sum_{k=1}^{N} |\hat{G}(\omega_k) - 1|$, with ω_k an excitation line is plotted as a function of the crest factor (= peak value/RMS value) for 1000 realizations in Fig. 3. For one third of the random multisines, the crest factor was actively pushed down using a crest factor minimizing algorithm [5],[8]. The crest factor minimization was early stopped to cover the interval [1.4-2.4]. Fig. 3. clearly shows that an odd multisine is doing significantly better than the consecutive one or the normally distributed noise excitation. This is due to the fact that it eliminates the even nonlinear contributions completely. The errors of the full multisine are also significantly smaller than those of the random excitation. The SNLS-terms dominate in the mean error. The theory guarantees that these terms disappear when the FRF is averaged over different realizations of the excitation (if the crest factor is not actively minimized, no preselection on basis of the crest factor is allowed). Moreover, under the same condition, the remaining G_{RLDS} (FRF obtained after an infinite number of averages=best linear approximation) is independent of the excitation signal that is used (Gaussian noise, consecutive multisine, odd multisine). This shows that we have all interest to select a signal with small SNLS contributions so that G_{RLDS} is obtained with a minimum number of averages. We advice to use the odd-multisine. It can also be remarked that both multisine errors seem to converge to the error of the binary signal. This can be understood by realizing that an optimized multisine has approximately a binary distribution. In Fig. 4 the mean absolute error with respect to the theoretical G_{RLDS} is shown for an odd random multisine as function of the crest factor. This

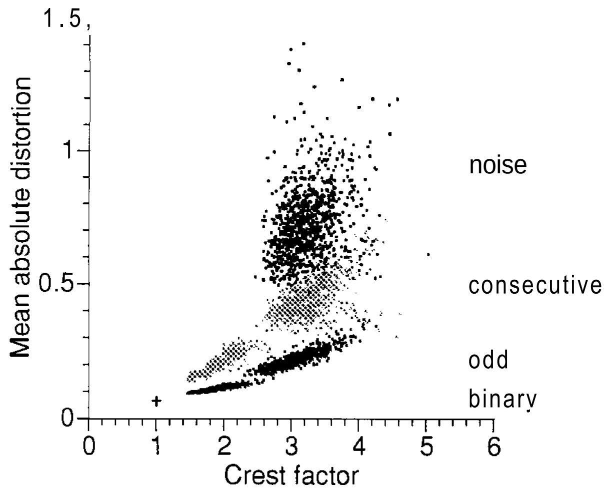

Figure 3:Mean absolute distortion for different excitation signals.

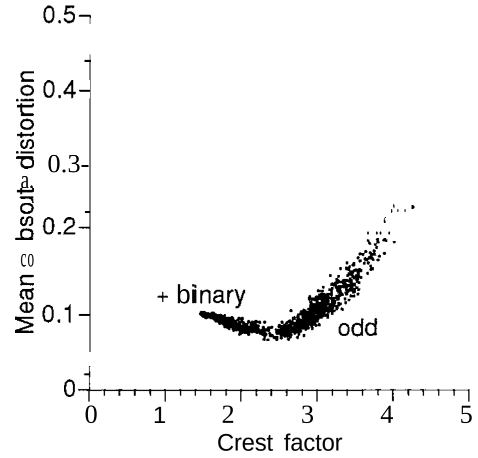

Figure 4:Mean absolute error on the measured best lint approximation ($|\hat{G}_{RLDS} - G_{RLDS}|$) for an odd multisine function of the crest factor

shows very clearly that the error first strongly decreases for decreasing crest factors, but then starts to grow again. The actual position of the minimum will depend on the nonlinear distortion. The increase of the error for small crest factors is due to the preselection effect. As mentioned before, for this subclass of multisines the RLDS is not guaranteed any more to converge to the overall RLDS (that also would be obtained with normally distributed noise excitations). For very small crest factors, these systematic deviations start to dominate the stochastic contributions.

Conclusion: measure the ‘best linear approximation’ using an odd multisine. If it is not possible to average, it is advised to select multisines with a small crest factor. If the results will be averaged over a series of realizations for the excitation signal, no crest factor minimization should be done.

B. Experimental verification

The previous conclusions are verified on the experimental setup. It was excited with Gaussian noise, a consecutive multisine ($f_k = kf_0$, $k = 20, 21, \ldots 143$), an odd ($k = 21, 23, \ldots 143$) and odd-odd multisine ($k = 21, 25, \ldots 141$), all with random phase and finally an odd multisine with minimized crest factor (about 1.4 1). For the random excitations 7 different realizations were measured, each time over 6 periods. On the basis of all these tests an estimate of the best linear approximation is made. This estimate is then used as a reference to compare the RLDS for each group of excitations. Two plots are made. The first one shows the mean (over the 7 realizations) of the absolute error to give an indication of the error level on each individual realization. The second shows the error on the averaged (over the 7 realizations) FRF. Here it can be seen that the FRF's converge all to same final result (at least within the uncertainty) as predicted by the theory, The error of the crest factor mini-

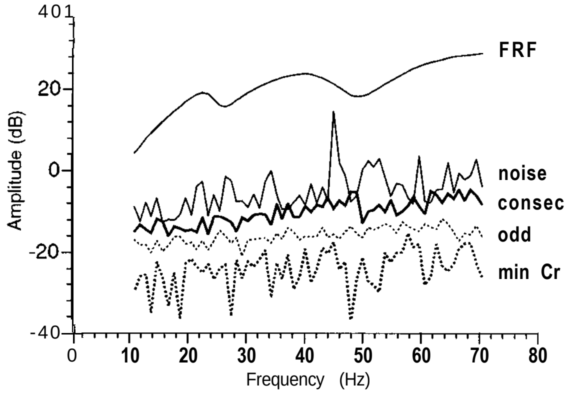

Figure 5:Mean error on the best linear approximation for the different classes of excitations.

mized multisine (MCMS) is smallest (although the average is dominated by the other signals since there was only one realization for the MCMS while there were 7 realizations for the others. Seemingly the stochastic errors do still dominate in this example.

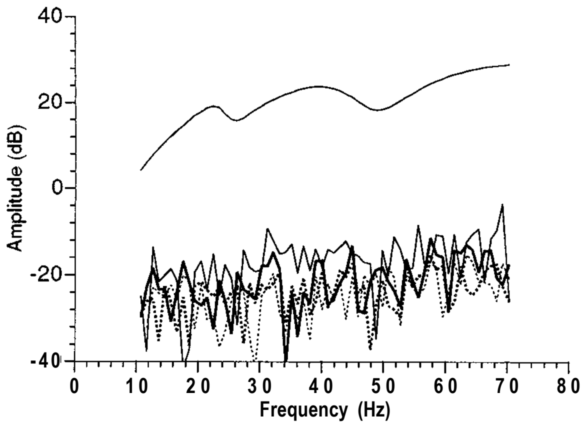

Figure 6:Error on the best linear approximation (average over 7 realization) for the different classes of excitations. See Fig. 5 for the legend.

Conclusion: In many cases with nonlinear distortions, the user wants to measure the best linear approximation to the nonlinear system. In this section it has be shown that this measurement can be obtained with minimal effort using odd random multisines having the same power spectrum as the signals that will be applied later on to the system. Compared to the classical (periodic) noise excitations, this approach has the major advantage that the stochastic nonlinear contributions are minimized so that the best linear approximation is reached within a given uncertainty with a minimum number of averages while the limit value itself is not affected by this choice.

VII. Conclusions

In this paper a simple introduction to FRF measurements in the presence of nonlinear distortions is given. First the nature of the error sources is explained. Next a simple methods was provided to detect, qualify and quantify the nonlinear distortions. Finally it was shown that the best linear approximation can be obtained with a minimum number of averages using odd multisines with minimized crest factor.

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