Reanalysis-based FEM for fuzzy uncertainty treatment in static structural analysis

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Abstract

Uncertainty modeling applied at different stages of the virtual prototyping process enhances the quality of the of the design in terms of physical validity. The Fuzzy FEM is a widely accepted numerical simulation tool to model imprecise data in early design stages. The Interval FEM, which forms the basis of the Fuzzy FEM, is based on the interval arithmetic approach or on black-box approaches such as global optimisation or vertex sampling. The limitation of the interval arithmetics applied in general IFE analysis is the large overestimation of the interval results. In order to make black-box approaches computationally less expensive, a novel reanalysis-based finite element method (ReFEM) is applied. This paper presents the ReFEM method for static structural analysis. By supplying analytical gradients, this approach is explicitly suited for optimisation-based black-box techniques. The computational benefits and general applicability in the context of Fuzzy FE analysis of the new approach is illustrated on a mid-sized plate problem.

1 Introduction

1.1 Non-deterministic (ND) modeling

In a virtual prototyping environment, based on numerical simulation tools, critical design objectives such as performance, reliability, robustness and safety are addressed. An important requirement for a reliable numerical simulation is physical trustworthiness. In order to enhance the credibility of the simulation tools, non-determinism has to be taken into account. Non-determinism is present in all stages of the design and it affects different aspects of the numerical simulation that is intended to represent the physical phenomena:

- Parametric non-determinism
  - different model parameters such as material properties, dimensions or sectional properties
  - boundary conditions such as clamping, contact, etc.
  - functional environment of the product such as temperature
  - loading of the system such as forces, gravity, heat, etc.
- Non determinism in the mathematical modeling
  - assumptions, simplifications, idealisations
  - lack of knowledge in the physical phenomena

- Numerical error in the modeling process
    - round-off errors
    - discretisation error

This paper focuses on parametric non-determinism, which based on the classification proposed by Oberkampf [1] is divided into two groups:

variability covers *the variation which is inherent to the modeled physical system or the environment under consideration*. Generally, this is described by a distributed quantity defined over a range of possible values. The exact value is known to be within this range, but it will vary from unit to unit or from time to time. Ideally, objective information on both the range and the likelihood of the quantity within this range is available. This type of non-determinism is also referred as *aleatory, irreducible, stochastic* or *due to chance* non-determinism.

uncertainty is *a potential deficiency in any phase or activity of the modeling process that is due to lack of knowledge*. The word *potential* stresses that the deficiency may or may not occur. This definition basically states that uncertainty is caused by incomplete information resulting from either vagueness, non-specificity or dissonance. Vagueness characterises information which is imprecisely defined, unclear or indistinct. It is typically the result of human opinion on unknown quantities. In literature, this type of non-determinism is also referred as *epistemic, reducible* or *subjective* non-determinism.

It is crucial to identify and characterize the critical ND parameters for each design stage. Ideally, the ND modeling techniques are matched with the different types of non-determinism affecting the virtual model [2]. The design evolution stages matched with the different ND approaches is presented in figure 1. As more information is gathered during the design evolution, the non-deterministic properties evolve from a coarse level of knowledge towards a fine level of knowledge, or from an interval data representation towards a probabilistic data representation. The profoundly elaborated and validated probabilistic methods are well accepted [3].

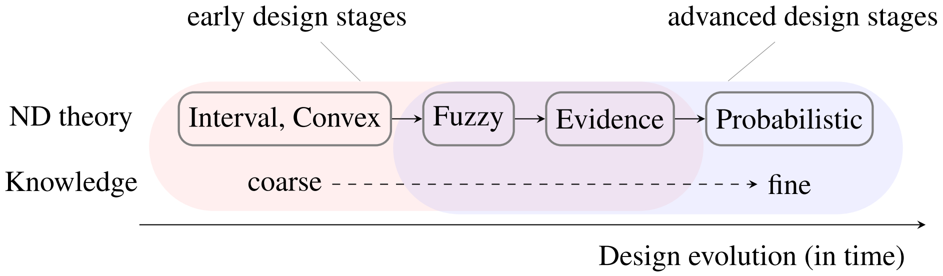


Figure 1: Evolution of the level of knowledge and ND approaches over the design time

These methods are well suited at an advanced design stage, where the full statistical data on the different non-deterministic parameters is available. In a preliminary or a conceptual design stage however, the lack of statistical data on the different imprecise parameters makes the value of the statistical non-deterministic models limited [4]. Assuming unrealistic probabilistic distributions may produce misleading results. Inferences drawn regarding safety, performance and reliability of mechanical systems based on assumed statistical data can be dangerous. At this stage, the use of the Fuzzy FE method (FFEM) can be complementary to the stochastic FE approaches. The FFEM is useful in a reliability framework with a possibilistic interpretation; furthermore FFEM is a valuable sensitivity and tolerance analysis and robust design optimisation tool [5].

1.2 Fuzzy FEM

The different numerical engineering tools can be extended to the concept of fuzzy numbers [6] using interval analysis (interval finite element method - IFEM). The most common implementation of the FFEM approach based on IFEM, is the α-cut strategy (see figure 2). The goal of the displacement based IFEM is to propagate the uncertainties on the input parameter space, represented by intervals, to the displacement output field. The static interval problem at each α membership level can be formulated as:

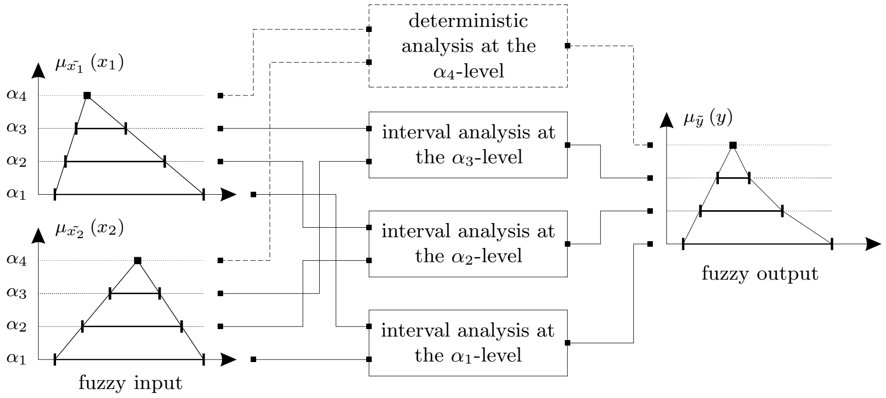


Figure 2: Fuzzy procedure

$$u_\alpha^I = \{u \mid u = f(x), x \in x_\alpha^I\} \tag{1}$$

where x_α^I and u_α^I the interval uncertain parameters, respectively, the interval displacement results at membership level α. $f(x)$ represents the FE analysis. It is not always possible to compute the exact interval solution, therefore a conservative approximation is sought.

1.3 Interval FEM implementations

The most straightforward implementation of the IFEM is the translation of the FE procedure to interval arithmetic. This approach however is practically of limited use due to the large overestimation. The category of black-box type approaches for IFEM implementation have the advantage of easy connection with existing FE codes. One of the most commonly used black-box type approach is the vertex method [7]. It is based on sampling of the uncertainty design space, and it is easily implemented. However, the approach requires monotonic input-output dependency in order to guarantee exactness. A different black-box strategy is based on global optimisation. This approach theoretically results in the exact hypercubic output field. However, it has the drawback of being computationally expensive and having an unpredictable performance. The performance of the strategy involving optimisation is influenced by the effectiveness of the optimisation procedure, the number of the uncertain parameters and the complexity of the displacement objective function.

In a more advanced black-box strategy developed by the authors [8], a parameter reduction scheme is applied in order to reduce the dimension and complexity of the optimisation problem. In the Reduced Optimisation (RO) technique the cost of the global optimisation is decreased by excluding the uncertain parameters with monotonic effect on the output. The parameter identification is based on a preliminary variation-pattern analysis. In a further development, the optimisation is accelerated using a surrogate model, which replaces the real response of the analysis. The Response Surface Method (RSM) developed by Box and Wilson [9], uses an approximation model of the expensive objective function based on only a few computed values [10, 11]. A promising strategy in the context of IFEM proves to be the RSM based on radial basis functions [12, 13] and central composite design (CCD). Based on the approximation error, this is implemented in an adaptive form. The reduction scheme is successfully combined with the RSM in the Reduced RSM [14] for efficient fuzzy static FE analysis.

In order to further increase efficiency of the IFEM and FFEM, the reanalysis-based finite element method (ReFEM) has been developed. The objective is to substantially reduce computational time of the core FE analysis. This paper presents the ReFEM, which supports optimisation-based black-box IFEM approaches, and illustrates the applicability on a space application. Section 2 presents the ReFEM with the two major components: fast system regeneration procedure based on explicit FE system matrix formulation (section 2.1) and fast reanalysis solver based on the preconditioned Conjugate Gradient method (section 2.2). Furthermore section 2.3 presents the extension of the proposed method towards system response gradient computation. Section 3 then presents the results of a fuzzy FE analysis of a rocket launcher component, using the novel ReFEM.

2 Reanalysis-based finite element method (ReFEM)

In the IFEM procedure, the core deterministic FE solver is integrated in the driver black-box procedures (global optimisation, vertex sampling,…). The classical black-box procedure is based on a readily available deterministic FE solver. Multiple calls of the core FE problem by the driver procedure generate the non-deterministic displacement field. The major computational cost of an interval or fuzzy analysis is the repeated FE system generation and solution. The motivation for the development of the ReFEM is to substantially reduce the computational burden of the core FE procedure. The newly developed ReFEM is coupled with the black-box approaches discussed above in order to form an efficient method for solving IFE and FFE problems (see figure 3). The improvement of the ReFEM rises from two parts, on the one hand from the fast system regeneration, and on the other hand from an efficient reanalysis solver:

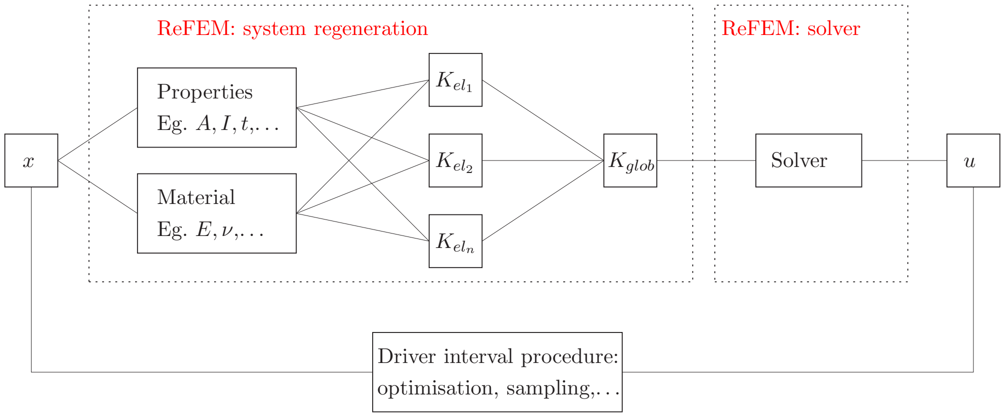


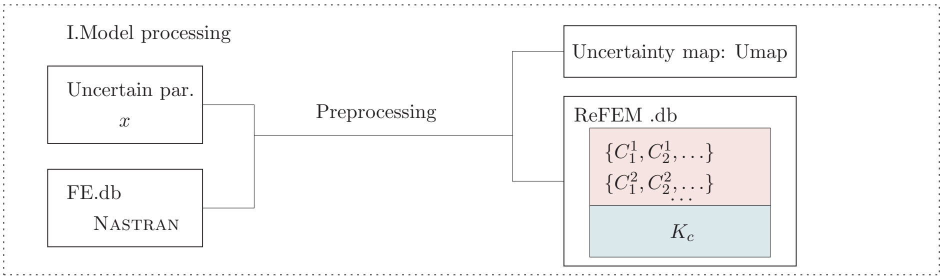
Figure 3: Integration of the ReFEM in the IFEM procedure

A. The system regeneration is based on an explicit element stiffness matrix formulation in terms of uncertain parameters. This formulation increases efficiency by avoiding the numerical integration of the element stiffness matrices. Furthermore, the regeneration procedure keeps track of the uncertainty distribution over the FE model, making 'on-demand' element generation possible. This means that only the part of the stiffness matrix that is changed from one design iteration to the next iteration is regenerated.

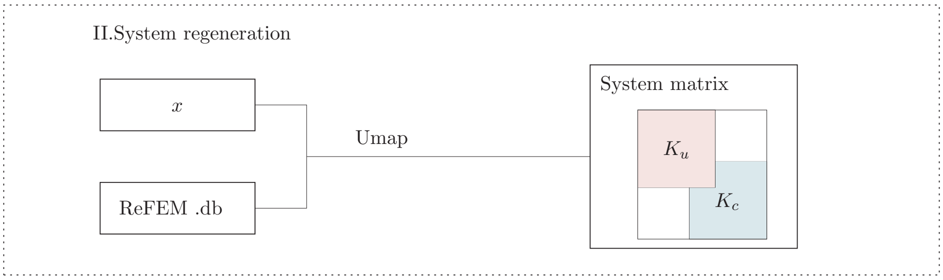
B. The solver is based on the following principle: obtain a fast solution of a modified FE system, by using the results from existing FE solutions. This reanalysis-based technique is founded on the preconditioned Conjugate Gradient (pCG) method [15, 16]. This iterative method is proved to be an efficient solver for linear positive definite symmetric systems arising in static structural FE analysis.

The following sections (2.1, 2.2) introduce in more detail both parts of the ReFEM. Furthermore, section 2.3 shows how the proposed parts support analytical gradient computation.

2.1 ReFEM: system regeneration



(a) I. Preprocessing step



(b) II. Regeneration step

Figure 4: The regeneration procedure

The system regeneration procedure is composed of two steps:

I. Preprocessing is performed at the beginning of the fuzzy process (figure 4(a)). At this step an uncertainty map (*Umap*) and a ReFEM database is generated. The uncertainty map links the uncertain parameters to the individual finite elements. The database contains the constant part of the stiffness matrix stored in matrix K_c. $C_i^j, (i = 1..nr_c)$ are nr_c constant matrices for each element j. These constant matrices are independent of the uncertain parameters and are calculated based on the explicit FE formulation concept.

II. Regeneration step is done repeatedly, driven by the driver fuzzy procedure. At each design iteration, the modification of the uncertain part of the stiffness matrix (K_u) is performed based on *Umap*. This modification is executed using the constant matrices C_i^j.

Explicit FE formulation is the basis for the regeneration procedure. In the classical implicit FE formulation, the stiffness matrix of a continuum element based on the isoparametric formulation is expressed as:

$$K_{el} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \left(J^{-1}B\right)^T D \left(J^{-1}B\right) |J| d\xi d\eta d\zeta = \sum_i \sum_j \sum_k w_{ijk} \left(J^{-1}B\right)_{ijk}^T D \left(J^{-1}B\right)_{ijk} |J|_{ijk} \quad (2)$$

In this expression, matrix D is the element elasticity matrix, and contains all material (eg. E-Young's modulus, ν-Poisson coefficient) and sizing (eg. t-plate thickness) parameters. Matrix B represents the strain-displacement relationship, which is specific to the element type and formulation. J stands for the Jacobian of the isoparametric transformation and includes geometry information of the element. ξ, η, ζ and w_{ijk} are the isoparametric coordinates, respectively, the weights of the Gauss integration. Considering the elements of matrix D constant over the integration domain, the explicit FE formulation is the result of an algebraic transformation of equation 2. In this formulation, the element stiffness matrix is expressed as:

$$K_{el} = \sum_{i=1}^{nr_c} f_i C_i \tag{3}$$

The functions f_i in expression 3 are rational expressions in the potential uncertain parameters. $C_i, (i = 1..nr_c)$ represent a number of nr_c constant matrices which store geometry data and finite element type dependent information but are independent of the material and sizing parameters. These parameters arising from the elasticity matrix D are considered as possible uncertain parameters. The structure of the constant matrices is similar to the structure of the stiffness matrix. Matrices C_i are computed in the preprocessing step using numerical integration. With the application of the coordinate transformation T on equation 3, matrices C_i are transformed into the global reference system as follows:

$$T^t K_{el} T = T^t \left(\sum_{i=1}^{nr_c} f_i C_i \right) T = \sum_{i=1}^{nr_c} f_i T^t C_i T \tag{4}$$

In the ReFEM process based on the explicit element formulation, the expensive numerical integration of matrix products is replaced by the summation of a few constant matrices weighted with simple algebraic functions. For optimisation-based black-box IFEM strategies, response gradients are required. Analytical gradient computation (section 2.3) requires the partial derivatives of the system matrix K. Derivatives $\frac{\partial K}{\partial x_i}$ are directly obtained based on the explicit FE formulation (equation 3).

Different element types have been transformed into the explicit form: rod, beam, membrane, plate and composite plate for both isotropic and orthotropic material description. As example, the explicit formulation is unfolded for both membrane and 3D plates.

4 node isotropic membrane element for 2D plane-stress analysis

The elasticity matrix of the isotropic membrane is given by

$$D = tE \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{5}$$

Matrix D has 3 different terms, therefore the algebraic transformation of equation 2 results in a sum of the 3 terms weighted by different constant coefficient matrices:

$$K_{el} = \frac{tE}{(1-\nu^2)} \acute{C}_1 + \frac{tE\nu}{(1-\nu^2)} \acute{C}_2 + \frac{tE(1-\nu)}{2(1-\nu^2)} \acute{C}_3 \tag{6}$$

The element stiffness matrix is expressed in the explicit form with $nr_c = 2$:

$$K_{el} = \frac{tE}{2(1-\nu^2)} C_1 + \frac{tE\nu}{2(1-\nu^2)} C_2, \text{ with } C_1 = 2\acute{C}_1 + \acute{C}_3 \text{ and } C_2 = 2\acute{C}_2 - \acute{C}_3 \tag{7}$$

4 node isotropic plate element for 3D analysis

The plate element is based on the combination of the thin-plate Kirchhoff bending theory and the membrane theory [19]. The membrane and bending elasticity matrices D_{memb} and D_{bend} are expressed in equations 8 and 9.

$$D_{memb} = tE \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{8}$$

$$D_{bend} = \left(\frac{t^3}{12} \right) E \frac{1}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \tag{9}$$

The plate stiffness matrix expressed in the explicit form with $nr_c = 4$, with 2 constant matrices C_{im} for the description of the membrane behaviour and 2 constant matrices C_{ib} for the bending, is determined as:

$$K_{el} = \frac{tE}{2(1-\nu^2)}C_{1m} + \frac{tE\nu}{2(1-\nu^2)}C_{2m} + \frac{t^3E}{2(1-\nu^2)}C_{1b} + \frac{t^3E\nu}{2(1-\nu^2)}C_{2b} \tag{10}$$

2.2 ReFEM: solver

Reanalysis solvers have been developed for analysis of structural modifications applied mainly in design optimisation [17, 18]. Reanalysis techniques are divided in literature in two groups: approximate and exact methods. Exact methods are based on the Sherman-Morrison Woodbury formulas and are limited to systems with low-rank modifications [20]. Approximate methods are suited for reanalysis of large modifications. A commonly referred approximate reanalysis method is the Combined Approximation (CA) [21], which combines the reduced basis method with a binomial series expansion. The preconditioned Conjugate Gradient method has been applied as reanalysis solver [23, 22] and has been shown to theoretically produce equivalent results to the CA [24, 25]. In this paper the pCG method is adopted to the solution of IFE problems. The advantage over the CA method is that the pCG method is computationally more efficient and has a built-in mechanism for evaluating the residual error after each iteration [23]. This greatly facilitates the implementation of a stop criterion for the iterative procedure, which makes possible controllable accuracy of the IFE and FFE procedures.

Preconditioned Conjugate Gradient method for the ReFEM

The pCG is an efficient iterative solver for symmetric positive definite linear systems resulting from linear static analysis. As for all Krylov subspace methods, search directions d_j are constructed by conjugation of the residuals [15, 16]. The solution of the static FE problem $Ku = F$ is expressed as:

$$u_i = u_0 + \sum_{j=1}^{i} \alpha_j d_j \tag{11}$$

The i^{th} iteration is an approximation of the exact solution, which is obtained by simple addition of the weighted sum (by scalar α_j) of the conjugated search directions d_j. The convergence rate of the iterative process depends on the condition number of the system matrix $\kappa(K)$ (see equation 12).

$$\|e_i\|_K \leq 2\Big[\frac{\sqrt{\kappa(K)}-1}{\sqrt{\kappa(K)}+1}\Big]^i \|e_0\|_K \tag{12}$$

The solution error $e_i = u_{exact} - u_i$ is reduced at a high rate with a favourable preconditioning and an optimal selection of the starting vector. The preconditioner matrix is chosen as the nominal stiffness matrix K_0. The starting vector u_o is the closest to the requested output u, chosen from the set of existing evaluations. The number of iterations is controlled by a stopping criterion based on a conservative estimate of the relative error norm (equation 13):

$$\frac{\|e_i\|}{\|u_{exact}\|} \leq \kappa(K_0^{-1}K)\frac{\|K_0^{-1}r_i\|}{\|u_0\|} \leq \epsilon \tag{13}$$

The residual norm of the preconditioned system $\|K_0^{-1}r_i\|$ is available at each iteration. $\kappa(K_0^{-1}K)$ and $\|u_0\|$ are computed in the preprocessing phase of the ReFEM. The criterion [26] presented in equation 13 guarantees that the relative error norm of the solution is smaller than the preselected value ϵ. The adopted reanalysis solver is applicable in case of both system and load uncertainties. In section 3, the proposed solver is compared to the commonly applied direct solution (DS) technique: Cholesky factorisation and forward-backward substitution (FBS).

2.3 Analytical gradient computation

Different black-box techniques for interval FE analysis are based on optimisation. Optimisation procedures well suited for smooth functions are gradient-based approaches eg. Newton's method, SQP and others. Conventionally, in the black-box approaches, gradients are supplied through finite differences. Analytical differences are more accurate, therefore more suited for optimisation, however analytical gradients are rarely available in structural FE codes. The new explicit FE formulation presented in section 2.1 offers straight support for analytical gradient computation. The partial derivative of the displacement vector with respect to uncertain parameter x_i is obtained by solving the linear equation 14, which is based on the differentiation of the system equation $Ku = F$.

$$K(x)\frac{\partial u}{\partial x_i} = \frac{\partial F}{\partial x_i} - \frac{\partial K}{\partial x_i}u(x), i = 1..nr_{unc} \tag{14}$$

Based on a set of linear equations (14) with size equal to the number of uncertain parameters (nr_{unc}), the gradient of the output vector u is obtained. Equation 14 is similar to the static system equation, with equal left hand side: K. Therefore, the proposed solution scheme presented in section 2.2 can be directly applied to this set of linear systems. The terms in equation 14 are summarized in table 1.

Table 1: Terms in the gradient equation

Term	Description	Computation
$K(x)$	stiffness matrix	system regeneration procedure
$\frac{\partial u}{\partial x_i}$	partial derivative of u	iterative system solver
$\frac{\partial F}{\partial x_i}$	partial derivative of F	differentiation
$\frac{\partial K}{\partial x_i}$	partial derivative of K	system regeneration procedure
$u(x)$	system response	ReFEM solver

The most demanding term being the partial derivative of the system matrix ($\frac{\partial K}{\partial x_i}$) is computed based on the explicit FE formulation presented in section 2.1. These terms are highly sparse matrices with size similar to the size of the system matrix K. The assembly of these matrices is performed in the same way as the assembly of the system matrix: individual element stiffness derivatives are added based on their connectivity. Equations 15, 16 and 17 represent the partial derivatives of the quadrilateral plate element stiffness matrix (equation 10).

$$\begin{aligned}
\frac{\partial K}{\partial E} &= \frac{t}{2(1-\nu^2)}C_{1m} + \frac{t\nu}{2(1-\nu^2)}C_{2m} + \frac{t^3}{2(1-\nu^2)}C_{1b} + \frac{t^3\nu}{2(1-\nu^2)}C_{2b} && (15)\\
\frac{\partial K}{\partial \nu} &= \frac{tE\nu}{(1-\nu^2)^2}C_{1m} + \frac{tE(\nu^2+1)}{2(1-\nu^2)^2}C_{2m} + \frac{t^3E\nu}{(1-\nu^2)^2}(1-\nu^2)^2C_{1b} + \frac{t^3E(\nu^2+1)}{2(1-\nu^2)^2}C_{2b} && (16)\\
\frac{\partial K}{\partial t} &= \frac{E}{2(1-\nu^2)}C_{1m} + \frac{E\nu}{2(1-\nu^2)}C_{2m} + \frac{3t^2E}{2(1-\nu^2)}C_{1b} + \frac{3t^2E\nu}{2(1-\nu^2)}C_{2b} && (17)
\end{aligned}$$

2.4 Summary

The steps of a ReFEM analysis required at each iteration of the driver black-box IFEM procedure are summarized in table 2.[1] At each optimisation iteration, a total number of $1 + nr_{unc}$ linear systems are solved with the ReFEM solver scheme.

Table 2: Summary of ReFEM analysis

Step	Description	Computation module	Result
1	stiffness matrix and gradient modification	system regeneration procedure	$K(x), \dfrac{\partial K}{\partial x_i}$
2	$Ku = F$ system solution	ReFEM solver	$u(x)$
3	$K(x)\dfrac{\partial u}{\partial x_i} = -\dfrac{\partial K}{\partial x_i}u(x)$ system solution	ReFEM solver	$\dfrac{\partial u}{\partial x_i}$

3 Numerical application

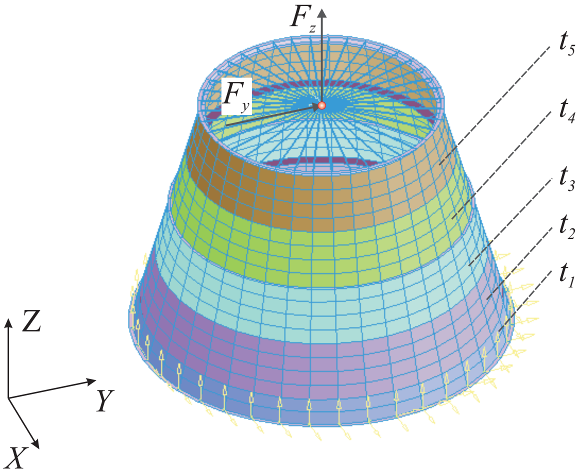


Figure 5: VEGA 1/2 interstage model

The demonstration application is the static analysis of a subpart of the small launcher VEGA (*ESA*), called VEGA interstage 1/2 (*Dutch Space*), presented in figure 5. The original FE model is coarsened to a reduced model with 6723 DOF's. The structure is subject to vertical and a horizontal nodal force, both with magnitude of $10kN$, and it is clamped at the lower side. The effect of 5 different sizing uncertain parameters (see table 3) on the static response at the force application node (ID=1204) is investigated. The massive reduction of the computational burden of a fuzzy analysis is illustrated with the application of the ReFEM combined with the global optimisation driver black-box procedure. The use of the FFEM is demonstrated as tolerance analysis tool.

In order to perform a fuzzy uncertainty analysis, the different parameters are subject to simple triangular membership functions with the base spanned by the range of the parameters and the tip being the nominal value of the parameters. The effect of the uncertainties defined on the input parameters is mapped to the displacements of node 1204 in both y and z directions. The main objective of this analysis is to demonstrate

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[1] Note that load uncertainties are not considered.

Table 3: Uncertainty definitions

Nr.	Description	Notation	Min $[mm]$	Nominal $[mm]$	Max $[mm]$
1	thickness	t_1	3	4	5
2	thickness	t_2	3	5	6
3	thickness	t_3	3	6	8
4	thickness	t_4	3	6	8
5	thickness	t_5	3	7	10

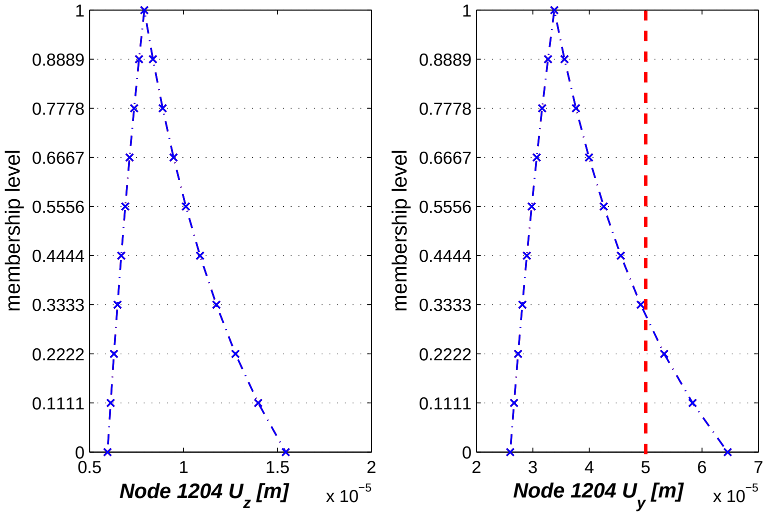

Figure 6: Fuzzy displacements at node 1204

the efficiency of the newly proposed ReFEM. Figure 6 presents the fuzzy results obtained with the classical FE solution process based on full system regeneration and the direct solver. The computational efficiency of the ReFEM for the fuzzy analysis of the model is presented in table 4. The CPU times[2] are based on 1261 crisp solutions for the DS with full system regeneration and 1080 crisp solutions for the ReFEM. The difference in number of function evaluations is due to the fact that the optimisation based on ReFEM is true gradient-based rather than finite difference-based. The total number of pCG iterations required by the ReFEM solver based on an error limit $\epsilon = 0.1\%$ is 1150, or 1.06 iteration for each function evaluation[3]. The low number of required iterations for the high accuracy on the displacement results are the effect of the small iteration steps applied in the optimisation procedure (SQP). Small iteration steps induce a small shift between the requested output and the starting vector u_0, therefore high convergence rate is obtained.

Table 4: Computational efficiency

CPU Time $[s]$	Direct	ReFEM	Relative $[\%]$
Generation	2617	46	1.7
Solution	52640	507	0.96
Preprocessing		332	
Total	55257	885	1.6

The fuzzy results as a large scale design sensitivity indication, show the simultaneous effect of the change in the range of the input parameters on the range of the displacements. A design criterion defined on node

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[2] 2x2GHz, 4GB

[3] partial derivatives are also accounted as function evaluations

1204 as $U_y \leq 0.05mm$ makes it possible to identify a range of allowable designs (see vertical dashed line on figure 6). The membership level which guarantees the design criterion is $\alpha \approx 0.3$. The corresponding design parameter ranges are presented in table 5.

Table 5: Feasible parameter ranges

Parameter	Lower $[mm]$	Upper $[mm]$
t_1	3.3	4.7
t_2	3.6	5.7
t_3	3.9	7.38
t_4	3.9	7.38
t_5	4.16	9.06

4 Conclusion

The lack of information in early design stages justifies the use of fuzzy analysis technique applied for uncertainty propagation. The FFEM for structural analysis is a widely accepted method to model imprecise data. One possible implementation of the FFEM is based on the α-cut strategy and interval FEM. Under special conditions interval analysis-based IFE problems result in acceptable overestimation, but for general problems the black-box type approaches are preferred.

This paper introduces the reanalysis-based FEM with the purpose of reducing computational cost of the repeated deterministic FE solutions that arise in a fuzzy FE analysis. The ReFEM increases the efficiency of the deterministic FE analysis in two ways, on the one hand with the fast system regeneration, and on the other hand with an efficient reanalysis solver. Moreover, this new approach facilitates the delivery of analytical gradients. The fast system regeneration procedure is based on the explicit FE formulation and makes 'on-demand' element stiffness matrix generation possible. The reanalysis solver is based on the efficient preconditioned Conjugate Gradient method. Optimal starting point selection, favourable preconditioning and a stop criterion to control accuracy makes this iterative procedure well suited for IFE and FFE analyses.

The performance and general applicability of the novel ReFEM is demonstrated on the fuzzy FE analysis of the reduced VEGA interstage 1/2 model. The computational cost of the FFEM based on the ReFEM is massively reduced compared to the fuzzy solutions based on the full system generation and direct system solver, while maintaining accuracy. In future research, the ReFEM can be extended towards stress analysis.

Acknowledgements

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