Fuzzy finite elements: combination of Guyan reduction and a new method to solve linear fuzzy system

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Abstract

In practice it is sometimes very difficult and in many cases even impossible to define correct and unique input data for structural mechanics applications. Fuzzy numbers can represent the uncertain input for those cases. As a consequence fuzzy arithmetic, based on the extension principle can be applied to solve finite element problems with uncertain parameters. Application of fuzzy arithmetic directly to the traditional techniques for the numerical solution of finite elements however turns out to be impracticable, especially solving systems of linear equations. Here we present a new method to solve systems of linear fuzzy equations combined with Guyan Reduction. Our conclusions are confirmed by a simple static problem.

1 Introduction

The finite element method is a well established and a widely used technique for the numerical simulation of different processes and phenomena in structures. The method was initially developed for structural mechanics applications in civil and mechanical engineering. Nowadays the applications area is however extremely wide with problems of heat transport, fluid flow, electromagnetism, ... The classical finite element method is a deterministic procedure: the structure is characterised by nominal values of geometrical and material properties. The two major steps of the method are the construction of a system of linear equations and solving the obtained system. The result of the analysis is also deterministic. In practice however it is very difficult and in many cases even impossible to define correct and unique input data. Fuzzy arithmetic may provide a solution for those cases. Different strategies are developed for the solution of the fuzzy finite element method. First, fuzzy procedures based on interval arithmetic can be used to find the equations for the solution, e.g. the vertex method of Hanss [3, 4]. An alternative approach based on global optimisation considers the deterministic finite element problem as a black-box goal function. Combining these two approaches, a hybrid approach has been developed [6]. Here we will first apply the technique of static condensation, thereafter the reduced system will be solved by a new method that is based on formulating the solution on the basis of parametric functions. First we will introduce the necessary notions in Section 2. In Section 3 we will formulate the problem used to illustrate our approach. Section 4 will explain a matrix reduction method, i.e. the static condensation. Thereafter we explain the method to solve systems of linear fuzzy equations on the basis of parametric functions. In the last section we will compare the different methods that can be used and draw the conclusion.

2 Preliminaries

First we recall some definitions concerning fuzzy numbers (see e.g. [5]). Let $A \in \mathcal{F}(\mathbb{R})$ (the class of fuzzy sets on the real line). Then A is convex if and only if

$$(\forall(x_1, x_2) \in \mathbb{R}^2)(\forall\lambda \in [0,1])(A(\lambda x_1 + (1-\lambda)x_2) \geq \min(A(x_1), A(x_2))).$$

If for $x \in \mathbb{R}$ it holds that $A(x) = 1$, then we call x a modal value of A. A unique modal value of A is denoted as $\operatorname{mod} A$.

The support of A is defined as

$$\operatorname{supp} A = \{x \mid x \in \mathbb{R} \text{ and } A(x) > 0\}.$$

A mapping f from $\mathbb{R}$ into $[0,1]$ is called upper-semicontinuous iff $(\forall\alpha{\in}[0,1])(f^{-1}([0,\alpha[) \in \tau_{|\cdot|})$, where $\tau_{|\cdot|}$ denotes the natural topology on $\mathbb{R}$ induced by the absolute value metric. As a consequence, when f is increasing: f is right-continuous iff f is upper-semicontinuous.

Definition 1 [5] *A fuzzy number is defined as a convex upper-semicontinuous fuzzy set on* $\mathbb{R}$ *with a unique modal value and bounded support. We denote by* $\mathcal{FN}$ *the set of all fuzzy numbers.*

From now on fuzzy numbers will be denoted by a lowercase letter with a tilde, e.g. $\tilde{a}$, and a vector of fuzzy numbers will be denoted as

$$\tilde{\mathbf{b}} = (\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_n)^T = \begin{pmatrix} \tilde{b}_1 \\ \tilde{b}_2 \\ \vdots \\ \tilde{b}_n \end{pmatrix},$$

where for any matrix A the transposed matrix is denoted as A^T. Sometimes we will denote the i-th component of $\tilde{\mathbf{b}}$ by $(\tilde{\mathbf{b}})_i$. Crisp numbers will be represented by a lowercase letter, e.g. a, and vectors of crisp numbers will be denoted as $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$.

A fuzzy number $\tilde{a}$ can be represented by its α-levels ($0 < \alpha \leq 1$):

$$\tilde{a}_\alpha = \{x \mid x \in \mathbb{R} \text{ and } \tilde{a}(x) \geq \alpha\}.$$

Here we define the α-level for $\alpha = 0$ as the support. Note that the α-levels of a fuzzy number are closed and bounded intervals (see Definition 1). One extends the support and the α-levels componentwise for vectors or matrices of fuzzy numbers. A triangular fuzzy number $A = (a/b/c)$ is a special case of a fuzzy number which membership function contains an increasing and decreasing linear part:

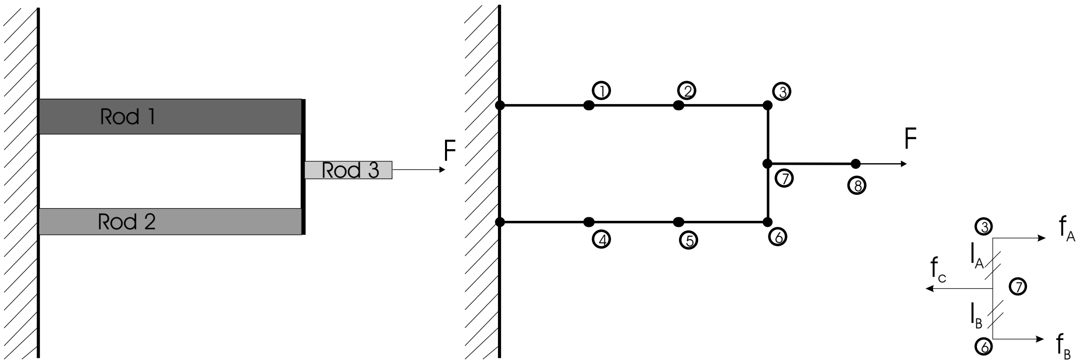
$$A(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{when } x \in \,]a,b] \\ \dfrac{c-x}{c-b} & \text{when } x \in [b,c[\\ 0 & \text{elsewhere.} \end{cases}$$

The arithmetic of fuzzy numbers is based on Zadeh's extension principle: let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers, then the sum of $\tilde{a}$ and $\tilde{b}$, denoted by $\tilde{a} \oplus \tilde{b}$, is given by, for all $z \in \mathbb{R}$,

$$(\tilde{a} \oplus \tilde{b})(z) = \sup_{z=x+y} \min(\tilde{a}(x), \tilde{b}(y)). \tag{1}$$

Analogous definitions follow for the fuzzy multiplication, subtraction and division. The fuzzy arithmetic based on the sup-min convolution (see (1)) can also be calculated by interval arithmetic applied to the α-levels: it is well-known that $(\tilde{a} \oplus \tilde{b})_\alpha = \tilde{a}_\alpha + \tilde{b}_\alpha$ and similarly for the fuzzy subtraction, multiplication and division.

Figure 1:



(a) Three rods, rigidly connected under tensile load

(b) Three rods divided in 8 nodes

(c) Loads on the multi-point constraint

3 Problem formulation

In this section we will formulate a rather simple static problem to provide a clear demonstration of the potential of our approach. Lets consider two parallel massless rods that have independent material properties, and a third rod is fixed rigidly to the two parallel rods (see Fig. 2(a)). The external loading consists of tensile forces F ($F = 1000N$) acting at the ends of the third rod. To determine the displacement of any cross section using the finite element method, the parallel rods are discretized into 2 elements and the third rod isn't discretized. The material properties for these three rods are displayed in the table:

Rod 1	$E_1 = 200 * 10^9(\frac{N}{m^2})$	$A_1 = 100 * 10^{-6}(m^2)$	$L_1 = 0.25(m)$
Rod 2	$E_2 = 69 * 10^9(\frac{N}{m^2})$	$A_2 = 75 * 10^{-6}(m^2)$	$L_2 = 0.25(m)$
Rod 3	$E_3 = 100 * 10^9(\frac{N}{m^2})$	$A_3 = 50 * 10^{-6}(m^2)$	$L_3 = 0.25(m)$

By L_i for $i \in \{1,2,3\}$, the length of each element of rod i is meant, A_i for $i \in \{1,2,3\}$ is the area of the cross section of rod i; f_A, f_B and f_C are the loads respectively on ③, ⑥ en ⑦ (see Figure 2(c)). The assembly for the element matrices in the finite element method results in the following system, given that $c_i = \frac{E_i A_i}{L_i}(\frac{N}{m})$ for $i \in \{1,2,3\}$:

$$\begin{pmatrix} 2c_1 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 2c_1 & -c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -c_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2c_2 & -c_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -c_2 & 2c_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -c_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & c_3 & -c_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & c_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_A \\ 0 \\ 0 \\ f_B \\ -f_C \\ F \end{pmatrix} \tag{2}$$

Since the displacements of nodes ③-⑥-⑦ (see Fig. 2(b)) are dependent of each other, we can rearrange our system by taking the following equations into account.

- The rigid connection between nodes ③-⑥ and ⑦ can be modelled by the equation $u_7 = \frac{1}{2}(u_3 + u_6)$;
- The equilibrium of loads creates the equation $f_A + f_B = f_C$;
- The equilibrium of moments leads to the equation $f_A l_A - f_B l_B = 0$

With this information some equations in the system (2):

$$\begin{cases} -c_1u_2 + c_1u_3 &= f_A \\ -c_2u_5 + c_2u_6 &= f_B \\ c_3u_7 - c_3u_8 &= -f_C \\ -c_3u_7 + c_3u_8 &= F \end{cases} \text{, are reformed to } \begin{cases} -c_1u_2 + (c_1 + \frac{1}{4}c_3)u_3 + \frac{1}{4}c_3u_6 - \frac{1}{2}c_3u_8 &= 0 \\ \frac{1}{4}c_3u_3 - c_2u_5 + (c_2 + \frac{1}{4}c_3)u_6 - \frac{1}{2}c_3u_8 &= 0 \\ -\frac{1}{2}c_3u_3 - \frac{1}{2}c_3u_6 + c_3u_8 &= F \end{cases}.$$

Therefore the system we have to analyse, is:

$$\begin{pmatrix} 2c_1 & -c_1 & 0 & 0 & 0 & 0 & 0 \\ -c_1 & 2c_1 & -c_1 & 0 & 0 & 0 & 0 \\ 0 & -c_1 & c_1 + \frac{1}{4}c_3 & 0 & 0 & \frac{1}{4}c_3 & -\frac{1}{2}c_3 \\ 0 & 0 & 0 & 2c_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & -c_2 & 2c_2 & -c_2 & 0 \\ 0 & 0 & \frac{1}{4}c_3 & 0 & -c_2 & c_2 + \frac{1}{4}c_3 & -\frac{1}{2}c_3 \\ 0 & 0 & -\frac{1}{2}c_3 & 0 & 0 & -\frac{1}{2}c_3 & c_3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ F \end{pmatrix}$$

When we solve this crisp system analytically, we obtain the solution:

$$\begin{cases} u_1 &= \frac{F}{2c_1} \\ u_2 &= \frac{F}{c_1} \\ u_3 &= \frac{3F}{2c_1} \\ u_4 &= \frac{F}{2c_2} \\ u_5 &= \frac{F}{c_2} \\ u_6 &= \frac{3F}{2c_2} \\ u_8 &= F(\frac{1}{c_3} + \frac{3}{4c_1} + \frac{3}{4c_2}) \end{cases} \tag{3}$$

When material uncertainty is considered, the Young's moduli are no longer crisp but can be modelled as fuzzy numbers $\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$. Here we assume a $\pm 5\%$ range around the crisp value of E_i for $i \in \{1, 2, 3\}$. Generally, measured experimental data shows a Gaussian distribution, however for simplicity we approximate these Gaussian curves with linear functions so that we achieve triangular fuzzy numbers. So we have $\tilde{c}_1 = (76 * 10^6 \ / \ 80 * 10^6 \ / \ 84 * 10^6)$, $\tilde{c}_2 = (1.97 * 10^7 \ / \ 2.07 * 10^7 \ / \ 2.17 * 10^7)$ and $\tilde{c}_3 = (1.9 * 10^7 \ / \ 2 * 10^7 \ / \ 2.1 * 10^7)$.

4 Static condensation

In this section we explain the principle of static condensation, also known as Guyan reduction. In 1965, Guyan [2] published a method to reduce the stiffness and mass matrices of the finite element method. Suppose the original system is given by:

$$Ku = f$$

with K the stiffness matrix, u the displacement vector and f the load vector. We can divide the displacement vector in boundary degrees of freedom (denoted with subscript t) and internal degrees of freedom (denoted with subscript o). The entire system is then partitioned:

$$\begin{bmatrix} K_{tt} & K_{to} \\ K_{ot} & K_{oo} \end{bmatrix} \begin{pmatrix} u_t \\ u_o \end{pmatrix} = \begin{pmatrix} f_t \\ f_o \end{pmatrix}$$

We can rewrite this system by its equations:

$$\begin{aligned} K_{tt}u_t + K_{to}u_o &= f_t \\ K_{ot}u_t + K_{oo}u_o &= f_o \end{aligned}$$

Out of the last equation we can extract u_o: $u_o = K_{oo}^{-1}(f_o - K_{ot}u_t)$. This can be filled in into the first equation so the reduced system is:

$$K_{ttred}u_{red} = f_{red}$$

with

- $K_{ttred} = K_{tt} + K_{to}G_{ot}$
- $f_{red} = f_t + G_{ot}^T f_o$

where G_{ot} is the static reduction matrix: $G_{ot} = -K_{oo}^{-1}K_{ot}$.

In our example, we only have to know the displacement of the nodes ③, ⑥ and ⑧, u_3, u_6 and u_8. The other displacements are internal degrees of freedom and aren't important for the solution. They can be omitted.

5 Solving systems of fuzzy equations using parametric functions

In this section we search for a solution of the matrix equation:

$$\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$$

for $\tilde{\mathbf{x}} = [\tilde{x}_k]_{n\times 1}$ where $\tilde{A} = [\tilde{a}_{ij}]_{n\times n}$ is a matrix with fuzzy numbers as entries and $\tilde{\mathbf{b}} = [\tilde{b}_k]_{n\times 1}$ is a vector of fuzzy numbers. Differently expressed,

$$\sum_{j=1}^{n} \tilde{a}_{ij}\tilde{x}_j = \tilde{b}_i, \qquad \text{for } 1 \le i \le n,$$

where fuzzy multiplication and addition based on the extension principle of Zadeh are used. Taking the α-levels of these equations we obtain systems of linear interval equations:

$$\begin{aligned}
&\sum_{j=1}^{n}[(\underline{a}_{ij})_\alpha, (\overline{a}_{ij})_\alpha][(\underline{x}_j)_\alpha, (\overline{x}_j)_\alpha] = [(\underline{b}_i)_\alpha, (\overline{b}_i)_\alpha], \text{for } 0 < \alpha \le 1 \text{ and } 1 \le i \le n,\\
&(\tilde{\mathbf{x}}_e)_i(x) = \sup\{\alpha \mid \alpha \in [0,1] \text{ and } x \in [(\underline{x}_i)_\alpha, (\overline{x}_i)_\alpha]\}, \qquad \forall x \in \mathbb{R}.
\end{aligned}$$

This solution is denoted as $\tilde{\mathbf{x}}_e$ as it is the exact solution of the system; when it is reentered into the system the equations are satisfied. However, these interval equations are hard to solve exactly and often $(\underline{x}_j)_\alpha$ and $(\overline{x}_j)_\alpha$ do not generate a fuzzy number (see [1]). Consequently the exact solution does not always exist and therefore the search for an alternative solution has a solid ground. Buckley and Qu [1] have already proposed a first solution. We follow their line of reasoning although the algorithm to find this solution can be optimized.

We require that the matrix $\tilde{A}$ of fuzzy numbers is regular in the sense that the matrix A^{-1} exists for all $a_{ij} \in \operatorname{supp}(\tilde{a}_{ij})$. Buckley and Qu [1] proposed to construct the set of all crisp solutions corresponding to the crisp systems composed with the elements in a certain α-level. They define the solution by, for all $\alpha \in]0,1]$,

$$\begin{aligned}
\Omega(\alpha) = \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n \text{ and } &(\exists A = [a_{ij}]_{n\times n} \in \mathbb{R}^{n\times n})(\exists \mathbf{b} = [b_k]_{n\times 1} \in \mathbb{R}^n)\\
&((\forall (i,j,k) \in \{1,2,\ldots,n\}^3)(a_{ij} \in (\tilde{a}_{ij})_\alpha \text{ and } b_k \in (\tilde{b}_k)_\alpha) \text{ and } A\mathbf{x} = \mathbf{b})\}
\end{aligned}$$

and for all $\mathbf{x} \in \mathbb{R}^n$,

$$\tilde{\mathbf{x}}_B(\mathbf{x}) = \sup\{\alpha \mid \alpha \in]0,1] \text{ and } \mathbf{x} \in \Omega(\alpha)\}. \tag{4}$$

We see that $\tilde{\mathbf{x}}_B$ is defined as a fuzzy set on $\mathbb{R}^n$ and not as a vector of fuzzy numbers. Therefore $\tilde{\mathbf{x}}_B(\mathbf{x})$ expresses to what extent the crisp vector $\mathbf{x}$ is a solution of the system of linear fuzzy equations $\tilde{A}\tilde{\mathbf{x}} = \tilde{\mathbf{b}}$.

We prefer to define a solution as a vector of fuzzy numbers to avoid information loss. Therefore we give a membership degree to every component of the solution vector and then $(\tilde{\mathbf{x}}_B)_i(x)$ expresses the degree to which x belongs to the fuzzy set $(\tilde{\mathbf{x}}_B)_i$, independent of $(\tilde{\mathbf{x}}_B)_j$, for all $j \neq i$. We thus define for all $x \in \mathbb{R}$ and for all $i \in \{1, 2, \ldots, n\}$,

$$(\tilde{\mathbf{x}}_B)_i(x) = \sup\{\alpha \mid \alpha \in]0,1] \text{ and } (\exists \mathbf{x} \in \Omega(\alpha))(x = x_i)\}, \tag{5}$$

where x_i denotes the i-th component of $\mathbf{x}$. This method is purely theoretical: in fact all crisp systems are solved. Even when we consider a finite number of α-levels, a lot of systems have to be solved, so the computation time will be large. In [8, 9] Vroman et al. proposed a practical algorithm to compute the solution. Instead of solving all these crisp systems, parametric functions of these solutions are determined.

5.1 Systems with one fuzzy coefficient

We first consider the case that we have to solve a system of linear fuzzy equations in which exactly one of the coefficients is a fuzzy number and the other coefficients are crisp numbers. Without loss of generality we may assume that $\tilde{a}_{11}$ is a fuzzy number. So we consider the following matrix equation:

$$\begin{pmatrix} \tilde{a}_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \tag{6}$$

where $\tilde{a}_{11}$ is a fuzzy number and $a_{ij} \in \mathbb{R}$, for all $(i,j) \in \{1,\ldots,n\}^2 \setminus \{(1,1)\}$, and $b_k \in \mathbb{R}$, for all $k \in \{1,\ldots,n\}$. In order to obtain the solution $\tilde{\mathbf{x}}_S$ of (6), we have to solve the crisp systems

$$A(a_{11})\mathbf{x} = \mathbf{b},$$

where

$$A(a_{11}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix},$$

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix},$$

for all $a_{11} \in]\underline{a}_{11}, \overline{a}_{11}[= \operatorname{supp}(\tilde{a}_{11})$. We can solve all of these systems through Cramer's rule thanks to the non-singularity of every crisp matrix $A(a_{11})$, for all $a_{11} \in \operatorname{supp}(\tilde{a}_{11})$. So we can write the solution for every component as a quotient of two determinants:

$$x_j = \frac{\begin{array}{ccccc} & & j & & \\ & & \downarrow & & \\ \multicolumn{5}{c}{\begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}} \end{array}}{\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}}.$$

The determinant of a matrix A is denoted as $|A|$. By expanding the determinants in the numerator and the denominator along the first row, we can write each component of the solution using parameters c_{1j}, c_{2j}, c_3 and c_4:

$$x_j = f_j(a_{11}) = \frac{a_{11}c_{1j} + c_{2j}}{a_{11}c_3 + c_4}. \tag{7}$$

Due to this result, every solution can be written using parametric functions with variable a_{11}. Note that c_{1j} and c_{2j} are dependent of j due to the fact that the j-th column in the numerator contains the components of $\mathbf{b}$. On the other hand, the denominator is the same for all $j \in \{1, \ldots, n\}$, so c_3 and c_4 are independent of j.

In [8] Vroman et al. proposed the following method to solve (6). First we compute the determinants of the matrices $A(\underline{a}_{11})$ and $A(\overline{a}_{11})$. The parameters c_3 and c_4 are obtained by solving the following system of linear crisp equations:

$$\begin{cases} |A(\underline{a}_{11})| = \underline{a}_{11}c_3 + c_4 \\ |A(\overline{a}_{11})| = \overline{a}_{11}c_3 + c_4. \end{cases}$$

We find

$$\begin{cases} c_3 = \dfrac{|A(\overline{a}_{11})| - |A(\underline{a}_{11})|}{\overline{a}_{11} - \underline{a}_{11}} \\ c_4 = |A(\overline{a}_{11})| - \overline{a}_{11}c_3. \end{cases} \tag{8}$$

We solve the crisp systems

$$A(\underline{a}_{11})\mathbf{x} = \mathbf{b}, \tag{9}$$

$$A(\overline{a}_{11})\mathbf{x} = \mathbf{b}, \tag{10}$$

and denote by $\underline{\mathbf{x}} = (\underline{x}_1, \ldots, \underline{x}_n)^T$ and $\overline{\mathbf{x}} = (\overline{x}_1, \ldots, \overline{x}_n)^T$ the solutions of (9) and (10) respectively. Then, for all $j \in \{1, \ldots, n\}$, we obtain c_{1j} and c_{2j} by solving the following system of crisp equations:

$$\begin{cases} \underline{x}_j|A(\underline{a}_{11})| = \underline{a}_{11}c_{1j} + c_{2j} \\ \overline{x}_j|A(\overline{a}_{11})| = \overline{a}_{11}c_{1j} + c_{2j}. \end{cases}$$

We obtain

$$\begin{cases} c_{1j} = \dfrac{\overline{x}_j|A(\overline{a}_{11})| - \underline{x}_j|A(\underline{a}_{11})|}{\overline{a}_{11} - \underline{a}_{11}} \\ c_{2j} = \overline{x}_j|A(\overline{a}_{11})| - \overline{a}_{11}c_{1j}. \end{cases} \tag{11}$$

Consequently, all possible solutions for the crisp systems $A(a_{11})\mathbf{x} = \mathbf{b}$, for all $a_{11} \in \operatorname{supp}(\tilde{a}_{11})$, can be obtained using (7). We define for all $j \in \{1, \ldots, n\}$ the fuzzy number $\tilde{x}_j$ by

$$(\tilde{x}_j)_\alpha = [\inf\{x \mid x = f_j(a_{11}) \text{ and } a_{11} \in (\tilde{a}_{11})_\alpha\}, \sup\{x \mid x = f_j(a_{11}) \text{ and } a_{11} \in (\tilde{a}_{11})_\alpha\}] \tag{12}$$

for all $x \in f_j(\mathrm{supp}(\tilde{a}_{11}))$, and

$$\tilde{x}_j(x) = 0,$$

for all $x \in \mathbb{R} \setminus f_j(\mathrm{supp}(\tilde{a}_{11}))$.

Moreover we see that the function f_j is monotone: for some real numbers $\hat{c}_{1j}$, $\hat{c}_{2j}$, $\hat{c}_3$ and $\hat{c}_4$, we calculate, for all $a_{11} \in \mathrm{supp}(\tilde{a}_{11})$,

$$\frac{\partial f_j}{\partial a_{11}}(a_{11}) = \frac{\hat{c}_{1j}\hat{c}_4 - \hat{c}_{2j}\hat{c}_3}{(a_{11}\hat{c}_3 + \hat{c}_4)^2},$$

where the denominator is always strictly greater than 0 because of the regularity of the matrix $\tilde{A}$. Hence f_j is monotone.

Therefore we only have to solve the crisp systems with the lower and upper limit of a certain α-level, denoted for example by $(\tilde{a}_{11})^L_\alpha$ and $(\tilde{a}_{11})^U_\alpha$ respectively, to find the α-level of the solution. We denote this solution by $\tilde{x}_I$:

$$((\tilde{x}_I)_j)_\alpha = [\min(f_j((\tilde{a}_{11})^L_\alpha), f_j((\tilde{a}_{11})^U_\alpha), \max(f_j((\tilde{a}_{11})^L_\alpha), f_j((\tilde{a}_{11})^U_\alpha)]$$

for all $x \in f_j(\mathrm{supp}(\tilde{a}_{11}))$, and $(\tilde{x}_I)_j(x) = 0$, for all $x \in \mathbb{R} \setminus f_j(\mathrm{supp}(\tilde{a}_{11}))$.

Note 1 Notice that for large n and m our method needs less computational effort than the method of Buckley and Qu, since they need to solve m crisp $n \times n$ systems. The total operation count (the number of additions, subtractions, multiplications and divisions) for the method of Buckley and Qu is equal to $\frac{m(4n^3+9n^2-7n)}{6}$. Because in this method we only have to calculate two determinants, solve two $n \times n$ systems and evaluate some expressions (details see table below), the operation count for our method equals $\frac{8}{3}n^3 + 4n^2 + (5m - \frac{14}{3})n + 3$.

2 determinants of $n \times n$ matrices	$\frac{4n^3+3n^2-n-6}{3}$
solving 2 $n \times n$ systems	$\frac{4n^3+9n^2-7n}{3}$
1 evaluation of (8)	5
n evaluations of (11)	$8n$
$n(m-2)$ evaluations of (7)	$5n(m-2)$
Total count:	$\frac{8}{3}n^3 + 4n^2 + (5m - \frac{14}{3})n + 3$

It is easy to see that for large n and m the method described above needs less computation time than the method of Buckley and Qu.

5.2 Systems with two fuzzy coefficients

Assume we have two fuzzy numbers $\tilde{a}_{11}$ and $\tilde{a}_{12}$, and assume that $a_{ij} \in \mathbb{R}$, for all $(i,j) \in \{1,\ldots,n\}^2 \setminus \{(1,1),(1,2)\}$ and $b_k \in \mathbb{R}$, for all $k \in \{1,\ldots,n\}$. Then we have the following system:

$$\begin{pmatrix} \tilde{a}_{11} & \tilde{a}_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}, \tag{13}$$

In order to obtain the solution $\tilde{\mathbf{x}}_I$ of (13), we have to solve the crisp systems

$$A(a_{11}, a_{12})\mathbf{x} = \mathbf{b},$$

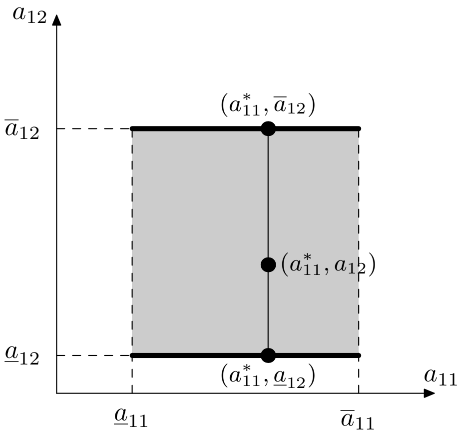


Figure 2: Solving systems with two fuzzy coefficients

for all $(a_{11}, a_{12}) \in \operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12})$, where

$$A(a_{11}, a_{12}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix},$$

and where $\mathbf{x}$ and $\mathbf{b}$ are defined similarly as in Subsection 5.1. In Figure 2 the grey area is the set $\operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12})$.

If we fix a_{12}, e.g. $a_{12} = \underline{a}_{12}$, then we have a system with only one fuzzy number $\tilde{a}_{11}$. So we find the solution in a similar way as in Subsection 5.1: for $j \in \{1, \ldots, n\}$, we find that

$$x_j = f_j(a_{11}) = \frac{a_{11}c_{1j} + c_{2j}}{a_{11}c_3 + c_4}.$$

Similarly as in Subsection 5.1 the values of c_{1j}, c_{2j}, c_3 and c_4, for $j \in \{1, \ldots, n\}$, are calculated. Thus, for any $a_{11} \in \operatorname{supp}(\tilde{a}_{11})$, the solution of the crisp system $A(a_{11}, \underline{a}_{12})\mathbf{x} = \mathbf{b}$ is obtained by calculating $\mathbf{x} = (f_1(a_{11}), f_2(a_{11}), \ldots, f_n(a_{11}))^T$. So, we have found the solution of the crisp systems corresponding to the points on the lower thick line in Figure 2. In a similar way, if we fix $a_{12} = \overline{a}_{12}$, then we can construct a function f_j' (with parameters c_{1j}', c_{2j}', c_3' and c_4'), for $j \in \{1, \ldots, n\}$, and we obtain for all $a_{11} \in \operatorname{supp}(\tilde{a}_{11})$ the solution of the system $A(a_{11}, \overline{a}_{12})\mathbf{x} = \mathbf{b}$ by calculating $\mathbf{x} = (f_1'(a_{11}), \ldots, f_n'(a_{11}))^T$. Thus the solution of the crisp systems corresponding to the points on the upper thick line in Figure 2 is obtained.

Now we fix arbitrarily $a_{11}^* \in \operatorname{supp}(\tilde{a}_{11})$ and let $a_{12} \in \operatorname{supp}(\tilde{a}_{12})$ vary. So, again, we obtain a system with only one fuzzy number, but this time the fuzzy number is $\tilde{a}_{12}$. Thus we are looking for the solution of the crisp systems corresponding to the points on the vertical thin line in Figure 2. Similarly as we did before for $\tilde{a}_{11}$, we can obtain the solution of the crisp system $A(a_{11}^*, a_{12})\mathbf{x} = \mathbf{b}$ as

$$x_j = f_j^{a_{11}^*}(a_{12}) = \frac{a_{12}c_{1j}^{a_{11}^*} + c_{2j}^{a_{11}^*}}{a_{12}c_3^{a_{11}^*} + c_4^{a_{11}^*}} \tag{14}$$

for all $j \in \{1, \ldots, n\}$. We find the parameters $c_3^{a_{11}^*}$ and $c_4^{a_{11}^*}$ by solving the system

$$\begin{cases} a_{11}^*c_3 + c_4 = \underline{a}_{12}c_3^{a_{11}^*} + c_4^{a_{11}^*} \\ a_{11}^*c_3' + c_4' = \overline{a}_{12}c_3^{a_{11}^*} + c_4^{a_{11}^*}. \end{cases}$$

We obtain

$$\begin{cases} c_3^{a_{11}^*} = \dfrac{a_{11}^*(c_3' - c_3) + c_4' - c_4}{\overline{a}_{12} - \underline{a}_{12}} \\ c_4^{a_{11}^*} = a_{11}^*c_3' + c_4' - \overline{a}_{12}c_3^{a_{11}^*}. \end{cases} \tag{15}$$

We have seen above that the solutions of the crisp systems

$$A(a^*_{11}, \underline{a}_{12})\mathbf{x} = \mathbf{b},$$
$$A(a^*_{11}, \overline{a}_{12})\mathbf{x} = \mathbf{b},$$

are given by $\underline{\mathbf{x}}^{a^*_{11}} = (f_1(a^*_{11}), \dots, f_n(a^*_{11}))^T$ and $\overline{\mathbf{x}}^{a^*_{11}} = (f'_1(a^*_{11}), \dots, f'_n(a^*_{11}))^T$ respectively. Then, for all $j \in \{1, \dots, n\}$, we obtain $c^{a^*_{11}}_{1j}$ and $c^{a^*_{11}}_{2j}$ by solving the following system:

$$\begin{cases} a^*_{11}c_{1j} + c_{2j} = \underline{a}_{12}c^{a^*_{11}}_{1j} + c^{a^*_{11}}_{2j} \\ a^*_{11}c'_{1j} + c'_{2j} = \overline{a}_{12}c^{a^*_{11}}_{1j} + c^{a^*_{11}}_{2j}. \end{cases}$$

We find that

$$\begin{cases} c^{a^*_{11}}_{1j} = \dfrac{a^*_{11}(c'_{1j} - c_{1j}) + c'_{2j} - c_{2j}}{\overline{a}_{12} - \underline{a}_{12}} \\ c^{a^*_{11}}_{2j} = a^*_{11}c'_{1j} + c'_{2j} - \overline{a}_{12}c^{a^*_{11}}_{1j}. \end{cases} \tag{16}$$

Consequently, all possible solutions for the crisp systems $A(a^*_{11}, a_{12})\mathbf{x} = \mathbf{b}$, for all $a_{12} \in \operatorname{supp}(\tilde{a}_{12})$, can be obtained using (14).

We now introduce for all $j \in \{1, \dots, n\}$ a function f_j on $\operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12})$ by, for $(a_{11}, a_{12}) \in \operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12})$,

$$f_j(a_{11}, a_{12}) = \begin{cases} f_j(a_{11}), & \text{if } a_{12} = \underline{a}_{12}, \\ f'_j(a_{11}), & \text{if } a_{12} = \overline{a}_{12}, \\ f^{a_{11}}_j(a_{12}), & \text{else.} \end{cases}$$

We define for all $j \in \{1, \dots, n\}$ the fuzzy set $\tilde{x}_j$ on $\mathbb{R}$ by

$$\begin{aligned} \tilde{x}_j(x) = \sup\{&\min(\tilde{a}_{11}(a_{11}), \tilde{a}_{12}(a_{12})) \mid \\ &(a_{11}, a_{12}) \in \operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12}) \text{ and } x = f_j(a_{11}, a_{12})\}, \end{aligned} \tag{17}$$

$$\begin{aligned} (\tilde{x}_j)_\alpha = [&\inf\{x \mid x = f_j(a_{11}, a_{12}) \text{ and } a_{11} \in (\tilde{a}_{11})_\alpha \text{ and } a_{12} \in (\tilde{a}_{12})_\alpha\}, \\ &\sup\{x \mid x = f_j(a_{11}, a_{12}) \text{ and } a_{11} \in (\tilde{a}_{11})_\alpha \text{ and } a_{12} \in (\tilde{a}_{12})_\alpha\}] \end{aligned} \tag{18}$$

for all $x \in f_j(\operatorname{supp}(\tilde{a}_{11}), \operatorname{supp}(\tilde{a}_{12}))$ and

$$\tilde{x}_j(x) = 0,$$

for all $x \in \mathbb{R} \setminus f_j(\operatorname{supp}(\tilde{a}_{11}), \operatorname{supp}(\tilde{a}_{12}))$. Finally, we define $\tilde{\mathbf{x}}_I = (\tilde{x}_1, \dots, \tilde{x}_n)^T$ and we call $\tilde{\mathbf{x}}_I$ the solution of the system (13).

Again we see that the function f_j is monotone in both arguments. Therefore we only have to calculate by means of the parametric functions the solution of the crisp systems obtained by all combinations of the lower and upper limit of a certain α-level to find the α-level of the solution:

$$\begin{aligned} ((\tilde{x}_I)_j)_\alpha = [&\min(f_j((\tilde{a}_{11})^L_\alpha, (\tilde{a}_{12})^L_\alpha), f_j((\tilde{a}_{11})^U_\alpha, (\tilde{a}_{12})^L_\alpha), f_j((\tilde{a}_{11})^L_\alpha, (\tilde{a}_{12})^U_\alpha), f_j((\tilde{a}_{11})^U_\alpha, (\tilde{a}_{12})^U_\alpha), \\ &\max(f_j((\tilde{a}_{11})^L_\alpha, (\tilde{a}_{12})^L_\alpha), f_j((\tilde{a}_{11})^U_\alpha, (\tilde{a}_{12})^L_\alpha), f_j((\tilde{a}_{11})^L_\alpha, (\tilde{a}_{12})^U_\alpha), f_j((\tilde{a}_{11})^U_\alpha, (\tilde{a}_{12})^U_\alpha)] \end{aligned}$$

Note 2 In [8, 9] it is shown that the total operation count for the method of Buckley and Qu, since they need to solve m^2 crisp $n \times n$ systems, is equal to $\frac{m^2(4n^3+9n^2-7n)}{6}$

The proposed method only requires the calculation of four $n \times n$ determinants, the solving of four $n \times n$ systems, $2n$ evaluations of (11) (n for c_{1j} and c_{2j}, and n for c'_{1j} and c'_{2j}), two evaluations of (8), $(m-2)n$ evaluations of (16) (for each $a^*_{11} \in \operatorname{supp}(\tilde{a}_{11}) \setminus \{\underline{a}_{11}, \overline{a}_{11}\}$ we have to evaluate (16) n times), $m-2$ evaluations of (15) and $((m-3)2+1)n$ evaluations of (14) (for each $a^*_{11} \in \operatorname{supp}(\tilde{a}_{11}) \setminus \{\underline{a}_{11}, \overline{a}_{11}, \operatorname{mod}\tilde{a}_{11}\}$ we have to evaluate (14) $2n$ times, for $\operatorname{mod}\tilde{a}_{11}$ only n times). In other words, we have to calculate four crisp $n \times n$ determinants, solve four crisp $n \times n$ systems, evaluate $m(n+1)$ expressions of the kind given by (8), (11), (15) and (16) and $((m-3)2+1)n$ expressions of the kind given by (14).

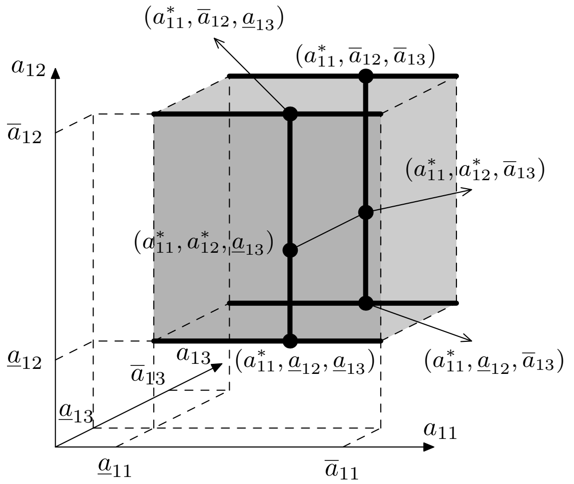


Figure 3: Solving systems with three fuzzy coefficients

4 determinants of $n \times n$ matrices	$\frac{8n^3+6n^2-2n-12}{3}$
solving 4 $n \times n$ systems	$\frac{8n^3+18n^2-14n}{3}$
$m(n+1)$ expression similar to (16)	$m(n+1)10$
$(m-3)2+1$ expression similar to (14)	$((m-3)2+1)5n$
Total count:	$\frac{16}{3}n^3 + 8n^2(20m - \frac{91}{3})n + 10m$

It is easy to see that for large n and m the method described above needs explicitly less computation time than the method of Buckley and Qu.

5.3 Systems with more than two fuzzy coefficients

For three fuzzy numbers, we can extend the procedure of Subsection 5.2 as follows. Assume we have three fuzzy numbers $\tilde{a}_{11}$, $\tilde{a}_{12}$ and $\tilde{a}_{13}$ and a_{ij} and b_k are crisp, for all $(i,j) \in \{1,\dots,n\}^2 \setminus \{(1,1),(1,2),(1,3)\}$ and $k \in \{1,\dots,n\}$.

We first fix $a_{13} = \underline{a}_{13}$ and use the method described in Subsection 5.2 to obtain the solution for all elements $(a_{11}, a_{12}, \underline{a}_{13})$ of the front face of the cube depicted in Figure 3. Then we fix $a_{13} = \overline{a}_{13}$ and we obtain similarly the solution for all elements $(a_{11}, a_{12}, \overline{a}_{13})$ of the back face of the cube. We then fix arbitrarily $(a^*_{11}, a^*_{12}) \in \operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12})$. We construct, for all $j \in \{1,\dots,n\}$, a function $f_j^{a^*_{11},a^*_{12}}$ with independent variable a_{13}. We find the corresponding parameters and then define a function f_j on $\operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12}) \times \operatorname{supp}(\tilde{a}_{13})$ by

$$f_j(a_{11}, a_{12}, a_{13}) = \begin{cases} f_j(a_{11}, a_{12}), & \text{if } a_{13} = \underline{a}_{13}, \\ f'_j(a_{11}, a_{12}), & \text{if } a_{13} = \overline{a}_{13}, \\ f_j^{a^*_{11},a^*_{12}}(a_{13}), & \text{else}, \end{cases}$$

where f_j is the parametric function defined on the front face and f'_j the parametric function defined on the back face of the cube. Finally the solution $\tilde{\mathbf{x}}_S$ is given by, for all $j \in \{1,\dots,n\}$ and $x \in \mathbb{R}$,

$$\begin{aligned} \tilde{x}_j(x) = \sup\{\min(\tilde{a}_{11}(a_{11}), \tilde{a}_{12}(a_{12}), \tilde{a}_{13}(a_{13})) \mid \\ (a_{11}, a_{12}, a_{13}) \in \operatorname{supp}(\tilde{a}_{11}) \times \operatorname{supp}(\tilde{a}_{12}) \times \operatorname{supp}(\tilde{a}_{13}) \\ \text{and } x = f_j(a_{11}, a_{12}, a_{13})\}, \end{aligned} \tag{19}$$

if $x \in f_j(\operatorname{supp}(\tilde{a}_{11}), \operatorname{supp}(\tilde{a}_{12}), \operatorname{supp}(\tilde{a}_{13}))$, and $\tilde{x}_j(x) = 0$, else.

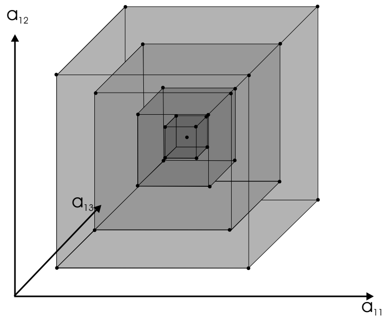


Figure 4: The necessary points for which the solution of the system is required ($K = 3$)

We see again that the parametric functions are monotone in all its arguments. Therefore we only have to calculate the solution obtained by all combinations of the lower and the upper limits of a certain α-level. First the parametric functions with variable a_{11} are obtained, i.e. the functions for $(a_{11}, \underline{a}_{12}, \underline{a}_{13})$, $(a_{11}, \overline{a}_{12}, \underline{a}_{13})$, $(a_{11}, \overline{a}_{12}, \overline{a}_{13})$, $(a_{11}, \underline{a}_{12}, \overline{a}_{13})$ with $a_{11} \in \mathrm{supp}(\tilde{a}_{11})$. With the obtained parameters the parametric functions with variable a_{12} and fixed $\underline{a}_{13}$ (resp. $\overline{a}_{13}$) i.e. the functions describing the front (resp. the back) face of the cube in Figure 4. Thereafter only the necessary parametric functions with variable a_{13} i.e. the functions that describe the lines such that $\tilde{a}_{11}(a_{11}) = \tilde{a}_{12}(a_{12})$ are calculated using the parameters of the front and back face of the cube. At last we only have to evaluate these parametric functions in the points (a_{11}, a_{12}, a_{13}) such that $\tilde{a}_{11}(a_{11}) = \tilde{a}_{12}(a_{12}) = \tilde{a}_{13}(a_{13})$. So we have to obtain m parametric functions for the front face (2 with variable a_{11} for the upper and lower line of the rectangle and $m - 2$ with variable a_{12} for $a_{11} \in]\underline{a}_{11}, \overline{a}_{11}[$), m for the back face. Furthermore for every $a_{11} \in]\underline{a}_{11}, \overline{a}_{11}[\setminus \{\mathrm{mod}(\tilde{a}_{11})\}$ there are two values of a_{12} for which the parametric functions with variable a_{13} has to be obtained. At last we obtain the parametric functions with variable a_{13} for $\mathrm{mod}(\tilde{a}_{11})$ and $\mathrm{mod}(\tilde{a}_{12})$. So, all together, we have to calculate 8 crisp $n \times n$ determinants, solve 8 crisp $n \times n$ systems, evaluate $(2m + 2(m - 3) + 1)(n + 1)$ expressions similar to (16), evaluate $(2^2(m - 3) + 1)n$ expressions similar to (14). In the same way, the method can be extended for K fuzzy coefficients. The advantage is of course that one can use earlier obtained parametric functions when a crisp coefficient in the system is replaced by a fuzzy coefficient. When the method is already applied for K fuzzy numbers, the obtained parametric functions span a compact K-dimensional set which can then be used to obtain the solution for $K + 1$ fuzzy coefficients since the method requires the parametric functions of two of those compact K-dimensional sets.

The total operation count of the method described in [8, 9] is equal to

$$= \frac{4}{3}2^K n^3 + 2^{K+1} n^2 + \frac{26}{3}2^K n + 33 * 2^{K-1} + 5 * 2^{K-1}(K(n + 1)(m - 3) - m) - 5n - 10$$

The total operation count for the method of Buckley and Qu, since they need to solve m^K crisp $n \times n$ systems is $\frac{m^K(4n^3+9n^2-7n)}{6}$. For large n, K and m the method described above needs less computation time than the method of Buckley and Qu. For example for a system of dimension 3 containing 3 fuzzy numbers, it is already advantageous to work with the new method if only 2 α-levels ($m = 3$) are considered.

In Table 1 and 2, the operation count between the method of Buckley and Qu and our proposed method is compared for a system with 8 fuzzy numbers where the dimension of the system and the number of α-levels is varying.

6 Combination of Guyan reduction and solving linear fuzzy systems

In the fuzzy finite element method we can benefit from both methods above. In this Section we wil discuss four possible approaches to find the solution for the fuzzy finite element method:

Table 1: Operation count for the proposed method ($K = 8$, m varying from 3 to 60, n varying from 2 to 1000)

$n \setminus m$	3	5	10	15	20	30	40	50	60
2	$9,52E+03$	$3,90E+04$	$1,13E+05$	$1,86E+05$	$2,60E+05$	$4,07E+05$	$5,54E+05$	$7,01E+05$	$8,49E+05$
10	$3,68E+05$	$4,79E+05$	$7,57E+05$	$1,04E+06$	$1,31E+06$	$1,87E+06$	$2,43E+06$	$2,98E+06$	$3,54E+06$
50	$4,28E+07$	$4,33E+07$	$4,46E+07$	$4,60E+07$	$4,73E+07$	$4,99E+07$	$5,25E+07$	$5,51E+07$	$5,77E+07$
100	$3,42E+08$	$3,43E+08$	$3,45E+08$	$3,48E+08$	$3,51E+08$	$3,56E+08$	$3,61E+08$	$3,66E+08$	$3,71E+08$
200	$2,73E+09$	$2,73E+09$	$2,74E+09$	$2,74E+09$	$2,75E+09$	$2,76E+09$	$2,77E+09$	$2,78E+09$	$2,79E+09$
300	$9,22E+09$	$9,22E+09$	$9,23E+09$	$9,24E+09$	$9,24E+09$	$9,26E+09$	$9,28E+09$	$9,29E+09$	$9,31E+09$
400	$2,18E+10$	$2,19E+10$	$2,19E+10$	$2,19E+10$	$2,19E+10$	$2,19E+10$	$2,19E+10$	$2,19E+10$	$2,20E+10$
500	$4,27E+10$	$4,27E+10$	$4,27E+10$	$4,27E+10$	$4,27E+10$	$4,27E+10$	$4,28E+10$	$4,28E+10$	$4,28E+10$
1000	$3,41E+11$	$3,41E+11$	$3,41E+11$	$3,41E+11$	$3,41E+11$	$3,41E+11$	$3,42E+11$	$3,42E+11$	$3,42E+11$

Table 2: Operation count for the method of Buckley and Qu ($K = 8$, m varying from 3 to 60, n varying from 2 to 1000)

$n \setminus m$	3	5	10	15	20	30	40	50	60
2	$5,90E+04$	$3,52E+06$	$9,00E+08$	$2,31E+10$	$2,30E+11$	$5,90E+12$	$5,90E+13$	$3,52E+14$	$1,51E+15$
10	$5,28E+06$	$3,14E+08$	$8,05E+10$	$2,06E+12$	$2,06E+13$	$5,28E+14$	$5,28E+15$	$3,14E+16$	$1,35E+17$
50	$5,71E+08$	$3,40E+10$	$8,70E+12$	$2,23E+14$	$2,23E+15$	$5,71E+16$	$5,70E+17$	$3,40E+18$	$1,46E+19$
100	$4,47E+09$	$2,66E+11$	$6,82E+13$	$1,75E+15$	$1,74E+16$	$4,47E+17$	$4,47E+18$	$2,66E+19$	$1,14E+20$
200	$3,54E+10$	$2,11E+12$	$5,39E+14$	$1,38E+16$	$1,38E+17$	$3,54E+18$	$3,53E+19$	$2,11E+20$	$9,06E+20$
300	$1,19E+11$	$7,08E+12$	$1,81E+15$	$4,65E+16$	$4,64E+17$	$1,19E+19$	$1,19E+20$	$7,08E+20$	$3,05E+21$
400	$2,82E+11$	$1,68E+13$	$4,29E+15$	$1,10E+17$	$1,10E+18$	$2,82E+19$	$2,81E+20$	$1,68E+21$	$7,21E+21$
500	$5,49E+11$	$3,27E+13$	$8,37E+15$	$2,15E+17$	$2,14E+18$	$5,49E+19$	$5,49E+20$	$3,27E+21$	$1,41E+22$
1000	$4,38E+12$	$2,61E+14$	$6,68E+16$	$1,71E+18$	$1,71E+19$	$4,38E+20$	$4,38E+21$	$2,61E+22$	$1,12E+23$

1. Solve the system analytically (see solution in Equation (3));
2. Apply static condensation on the parametric system and solve system of fuzzy equations using parametric functions;
3. Apply static condensation on the numerical system and solve system of fuzzy equations using parametric functions;
4. Solve the system without any preprocessing;

The first approach is of course the most accurate: working with the parameters has the advantage that the dependencies between arguments of elementary operations are taken into account so no artificial uncertainty is added as in the case of fuzzy operations. In practice however, solving the system analytically can be very hard or only the numerical data of the system is available. When static condensation is applied on the parametric system and thereafter the proposed method to solve systems of fuzzy equations is used (2), we aim at as less as possible multiple appearances parameters in the system. In the case no multiple appearance of parameters are present in the system, there is no artificial uncertainty in the result. In practical applications it is not always easy to condense the parametric system because for example the system is very big or only the numerical data is available. By applying the Guyan reduction on the numerical system (3), the dependencies between the arguments in the computation of the reduction are not taken into account, so artificial uncertainty is added. But it is still avantageous to do the reduction:

- There is less artificial uncertainty added to the result than when the system is solved directly without any preprocessing.
- The computation time of the method to solve systems of linear fuzzy equations grows exponentially with the number of fuzzy entries in the system. In the case of a condensed system the profit on the computation time is spectacular.
- The rounding errors also pile up when the system is bigger because earlier computed subresults are used to obtain the results of the next level. The smaller the number of fuzzy entries, the smaller the amount of rounding errors is.

So the biggest advantage is achieved when the Guyan reduction is applied on the parametric system. Thereafter the condensed system is solved using parametric functions. These conclusions are confirmed by our

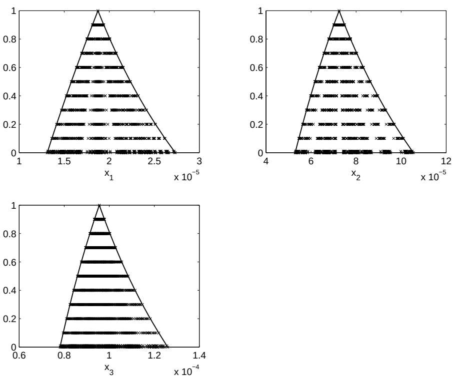


Figure 5: Displacements computed after parametric condensation (11 α-levels)

problem with 3 rods. The internal degrees of freedom, that can be omitted, are u_1, u_2, u_4 and u_5. Consequently we have:

$$K_{tt} = \begin{pmatrix} c_1 + \frac{c_3}{4} & \frac{c_3}{4} & -\frac{c_3}{2} \\ \frac{c_3}{4} & c_2 + \frac{c_3}{4} & -\frac{c_3}{2} \\ -\frac{c_3}{2} & -\frac{c_3}{2} & c_3 \end{pmatrix} \; ; \; K_{to} = K_{ot}^T = \begin{pmatrix} 0 & -c_1 & 0 & 0 \\ 0 & 0 & 0 & -c_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \; ;$$

$$K_{oo} = \begin{pmatrix} 2c_1 & -c_1 & 0 & 0 \\ -c_1 & 2c_1 & 0 & 0 \\ 0 & 0 & 2c_2 & -c_2 \\ 0 & 0 & -c_2 & 2c_2 \end{pmatrix} .$$

For the reduction we have:

$$G_{ot} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 \end{pmatrix}$$

So the condensed parametric system is:

$$\begin{pmatrix} \frac{c_1}{3} + \frac{c_3}{4} & \frac{c_3}{4} & -\frac{c_3}{2} \\ \frac{c_3}{4} & \frac{c_2}{3} + \frac{c_3}{4} & -\frac{c_3}{2} \\ -\frac{c_3}{2} & -\frac{c_3}{2} & c_3 \end{pmatrix} \begin{pmatrix} u_3 \\ u_6 \\ u_8 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ F \end{pmatrix}$$

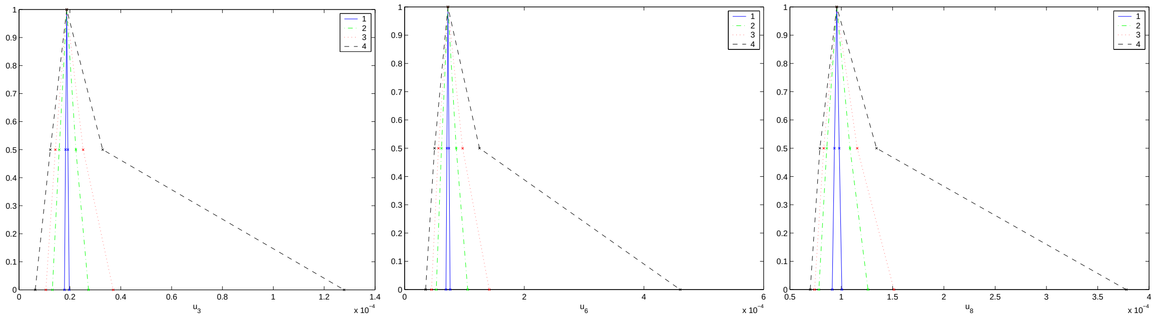
This condensed parametric system is then solved by the method proposed in Section 5. The result, for 11 α-levels is plotted in Figure 5.

In Figure 6 we make the comparison between the four proposed approaches for three α-levels, $\alpha \in \{0, 0.5, 1\}$, We see that the second approach of reducing the parametric system before solving has a little bit more uncertainty than the exact analytical solution (1), because there are multiple appearances of parameters in the condensed parametric system (and rounding errors in the numerical solution of the system) and in the analytical solution there are no multiple appearances. On the other hand, when we apply the Guyan reduction on the numerical system (3), there is more uncertainty due to the artificial uncertainty added during the static condensation by ignoring dependencies between the entries of the matrices. However when the system isn't preprocessed (4), a lot of artificial uncertainty is added to the result and the computation time is very large.

7 Conclusion

In conclusion, the combination of the Guyan reduction and solving linear fuzzy systems using parametric functions is very advantageous, as well as in reducing the artificial uncertainty as in reducing the computation

Figure 6: Displacements (3 α-levels)



time. The exact solution found by solving the system analytically isn't possible in real applications. The best strategie is to reduce the parametrical system if that is possible, and then solve the system numerically with the proposed method. If the parametrical system is unknown, Guyan reduction is still very advantageous to apply, not only in adding less artificial uncertainty to the solution but also in saving computation time.

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