Non-probabilistic approaches for non-deterministic dynamic FE analysis of imprecisely defined structures

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Abstract

The objective of this paper is to give a survey on the use of non-probabilistic methods for non-deterministic numerical analysis. It consists of a general theoretical discussion on the use of non-deterministic concepts in numerical analysis, an overview of the basic properties of possible implementations of the non-probabilistic finite element code, and finally a discussion on their possible applications in a typical product design process. In the first part of the paper, a classification of different types of non-deterministic properties is presented. Based on the nature of these different classes of model properties, it is discussed to what degree each of these fits in the framework of either a probabilistic or a non-probabilistic concept.

In the second part of the paper, an overview of possible implementations of the non-probabilistic finite element procedure is presented. It is shown how the deterministic finite element procedure is translated to a non-probabilistic counterpart. The paper then focuses on the application of these non-probabilistic concepts for design purposes. The use of the probabilistic and non-probabilistic approaches is projected into a typical design process.

1 Introduction

The continuous exponential growth of computational capabilities of modern computers clearly has an impact on the use of the numerical simulation techniques like the finite element (FE) method for engineering purposes. On the one hand, this evolution enables the use of very detailed and hence large FE models. Simultaneously, this evolution has paved the way for a number of computationally intensive analysis techniques derived from the classical FE technique (non-linear analysis, multiphysics...). In this context, the non-deterministic FE analysis has become a very important issue in the search for valuable utilisation of the available computational resources. Over the past decade, the probabilistic FE analysis has gained a large popularity in this area.

Recently, a number of non-probabilistic approaches for non-deterministic analysis are emerging. The Interval FE (IFE) analysis is based on the interval concept for the description of non-deterministic model properties, and so far has been studied only on an academic level [1, 2, 3]. The Fuzzy FE (FFE) analysis is basically an extension of the IFE analysis, and has been studied in a number of specific research domains: static structural analysis [4, 5], dynamic analysis [6, 7, 8], geotechnical engineering [9, 10], multibody kinematics [11], steady-state analysis of rolling [12], analysis of smart structures [13] and analysis of fiber-reinforced composite materials [14]. The numerical procedures developed for the non-probabilistic approaches are all strongly influenced by the specific properties of the analysed physical phenomenon, and only academic examples with very limited size and complexity are considered.

The non-probabilistic approaches broaden the possibilities, but simultaneously complicates the choice for the analyst. Therefore, it is important to study to what extend they can be an alternative in the areas where probabilistic analysis has become the standard. The growing interest for non-probabilistic methods for non-deterministic numerical analysis mainly originates from criticism on the credibility of probabilistic analysis when it is based on limited information. Especially when extremely high reliabilities are analysed based on numerical models, design engineers often remain very sceptic regarding the trustworthiness of the numerical predictions. The recent development of the non-probabilistic approaches stems from the argumentation that this lack of credibility is always present in probabilistic analysis results, but generally remains unaccounted for. It is argued that the non-probabilistic concepts could be more appropriate to model certain types of non-deterministic information, resulting in a better representation of the simulated non-deterministic physical behaviour. Also, it is believed that a full probabilistic description of a non-deterministic event is not always required. Especially in early design stages, when objective probabilistic information often is not available, non-probabilistic concepts are considered to be of great value. It is the aim of this paper to critically review this argumentation, and to study to what extent the non-probabilistic methods can be considered as useful alternatives to the existing probabilistic approach.

Section 2 contains a general theoretical discussion on the use of non-deterministic concepts in numerical analysis. This discussion is founded on different possible interpretations of the information represented by non-deterministic properties, summarised in a classification in section 2.1. Section 3 describes how the non-probabilistic concepts can be translated to a functional FE methodology. Based on this mathematical background, this section finally discusses how non-probabilistic analysis can be of use in a design process.

2 General aspects of non-determinism in numerical analysis

2.1 Sources of numerical non-determinism in general FE modelling

2.1.1 Definitions

In literature, the use of the terminology *error*, *uncertainty* and *variability* is not unambiguous. Different researchers apply the same terminology but the meaning attached to these is rather inconsistent. This necessitates a profound clarification of the terminology for each publication which treats uncertainties. This work does not propose a new terminology, but applies the terminology proposed by OBERKAMPF [15]. Some additional nuances are, however, necessary in order to enable clear distinction between probabilistic and non-probabilistic quantities in the remainder of this chapter.

The term *variability* covers *the variation which is inherent to the modelled physical system or the environment under consideration*. Generally, this is described by a distributed quantity defined over a range of possible values. The exact value is known to be within this range, but it will vary from unit to unit or from time to time. Ideally, objective information on both the range and the likelihood of the quantity within this range is available. Some literature refers to this variability as *aleatory uncertainty* or *irreducible uncertainty*, referring to the fact that even when all information on the particular property is available, the quantity cannot be deterministically determined. Typical examples of variability are:

- a property of a design which is subject to manufacturing tolerances
- environmental effects on a model (temperature, humidity, . . .)
- properties of non-uniform materials
- identifiable disturbances in operating conditions

An *uncertainty* is *a potential deficiency in any phase or activity of the modelling process that is due to lack of knowledge*. The word *potential* stresses that the deficiency may or may not occur. This means that there may be no deficiency even though there is some lack of knowledge, i.e. , when the numerical model of the phenomenon happens to be correct rather by chance than due to exact knowledge. This definition basically states that uncertainty is caused by incomplete information resulting from either vagueness, nonspecificity or dissonance [16]. Vagueness characterises information which is imprecisely defined, unclear or indistinct.

It is typically the result of human opinion on unknown quantities ("*the density of this material is around* x"). Nonspecificity refers to the availability of a number of different models that describe the same phenomenon. The larger the number of alternatives, the larger the nonspecificity. Dissonance refers to the existence of conflicting evidence of the described phenomenon, for instance when there is evidence that a quantity belongs to disjoint sets. Possibly, limited objective information is available, for instance when a range of possible values is known. In most cases, however, information on uncertainties is subjective and based on some expert opinion. Others in literature refer to this uncertainty as *reducible*, *epistemic* or *subjective uncertainty*. Typical examples of uncertainties are:

- non-rigid models for boundary conditions
- simplified models for joints
- models for material damping
- unpredictable model changes due to ageing, loading, ...

An *error* is defined as *a recognisable deficiency in any phase of modelling or simulation that is not due to lack of knowledge*. The fact that the error is recognisable states that it should be identifiable through examination, and as such is not caused by lack of knowledge. This means that the error could be avoided by an alternative approach which is known to be more accurate, but which is possibly limited in practical applicability by computational cost or other practical considerations. A further distinction between *acknowledged* and *unacknowledged* errors is possible. Unacknowledged errors are blunders or mistakes, for instance where the analyst tries to model one phenomenon but as a result of human error, applied the wrong governing equations. These unacknowledged errors cannot be corrected. An example of an acknowledged error is the error associated with the conversion of partial differential equations into discrete equations using the FE methodology. Other typical examples of acknowledged errors in FEM are:

- the use of linear models
- the use of partial differential equations for the description of the analysed phenomenon
- the representation of the displacement inside an element by a polynomial shape function

It is clear that the acknowledged errors are all inherent to the FE analysis methodology as for any numerical analysis tool which tries to describe physical reality. This means that remedying these errors implies an alternative approach to the classical FEM principles. This, however, lies not within the scope of this paper.

Figure 1 summarises the definitions in this section with their main characteristics in the context of the FE methodology.

2.1.2 Discussion and extension of the definitions

The definitions of uncertainty and variability above are rather straightforward and comprehensible. However, they are not mutually exclusive, since a variability could be subject to lack of knowledge when information on its range or likelihood within the range is missing. This is for instance the case for every design dimension subject to tolerances, but without further specification of manufacturing process or supplier. The tolerances represent the bounds on the feasible domain, but there is no information on the likelihood of the possible values within these bounds. Consequently, because there is a lack of knowledge, such a variability is also an uncertainty. It is referred to here as an *uncertain variability*. Some vague knowledge may be available ("*the mean value is approximately* x") but also nonspecificity may play an important role in the uncertainty, for instance in choosing an appropriate model to describe a random quantity. Opposed to the uncertain variability, a *certain variability* refers to a variability the range and likelihood of which are exactly known.

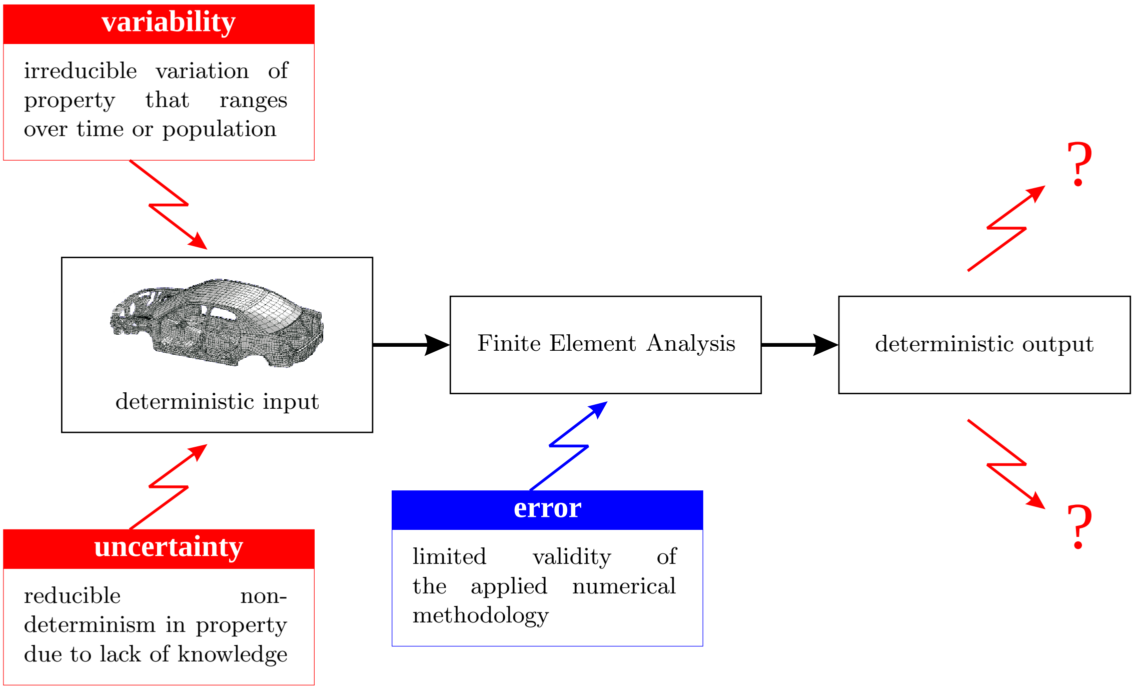

Figure 1: Occurrence of variabilities, uncertainties and errors in the FE procedure

On the other hand, it appears logical to state that every property in a numerical model corresponding to a physical quantity is a variability, since it will eventually have a range of possible values and a likelihood inside this range in the physical model. This argumentation implies that all uncertainties are also variabilities. In practice, however, the majority of model properties are implemented as constant deterministic values in the numerical model. Though they are subject to variation, the influence of their variability on the analysis result is considered to be negligible. Often, uncertainties refer to a possible lack of knowledge in these deterministic properties. This type of uncertainty is referred to as *invariable uncertainty*. Note that *invariable* in this case does not mean that the property cannot change over different analyses. According to the definition of uncertainty, it will change when information that decreases the amount of uncertainty is acquired. The invariable uncertainties typically occur in model properties for model parts that are difficult to describe numerically, but considered constant in the final physical product (connections, damping, . . .). Other examples are design properties which have negligible variability but which are not defined exactly in an early design stage. Figure 2 gives a graphical illustration of the proposed subdivision of the definitions for uncertainty and variability.

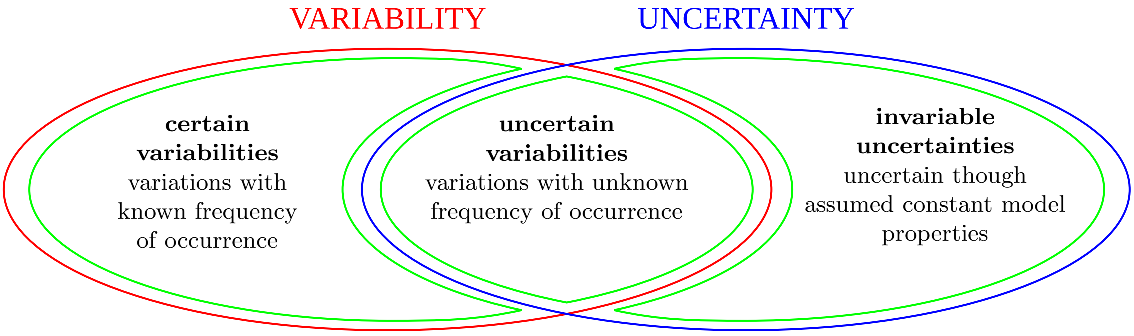

Figure 2: Classification of variabilities and uncertainties in numerical modelling

2.1.3 Example

The above definitions give a clear distinction between the different classes of non-deterministic model properties. In order to do a meaningful analysis, it is very important to determine the right class of the non-deterministic properties present in the treated problem. The following small example illustrates that this classification is not always obvious, and can be influenced by the intention of the non-deterministic analysis.

Consider of a tank design, which is designed to carry a prescribed amount of fuel. Suppose that we want to assess the structural design of this tank through numerical analysis, using the amount of fuel in the tank as non-deterministic parameter. In other words, we want to analyse the effect of the amount of fuel in the tank on its structural behaviour. In the most conventional interpretation of this problem, the purpose is to account for all possible uses of the tank, from being empty to completely filled. In that case, the mass of the fuel in the tank can be interpreted as a variability, since there clearly will be a variation over time or from unit to unit in the amount of fuel the tank contains. This means that the analysis indeed should focus on the effect of this variation in fuel mass between the absolute minimum and the absolute maximum.

On the other hand, the problem is totally different when we want to analyse the same tank when it is designed for a single mission, e.g. , as a fuel supply tank in a space mission. In that case, the difference between the actual carried amount of mass and the amount the tank is designed for can be expected to be very small. The amount of fuel now no longer represents a variability. It could however be useful to analyse its influence on the design, for instance to assess the robustness of the design. In that case, the fuel mass becomes an invariable uncertainty, the range of which can be chosen by the analyst according to his own expert knowledge or preference.

2.2 Numerical concepts for non-deterministic numerical modelling

Historically, the introduction of the non-probabilistic approaches for non-deterministic analysis has initiated a profound discussion in literature. On one side, some claim that the probabilistic approach is only a subcategory of a more universal non-probabilistic approach. Therefore, the latter would represent a more unified approach for non-deterministic analysis. On the other side, some argue that probabilistic methods are able to model anything the non-probabilistic approach can. The goal of this section is not to choose either side in this discussion, but merely to review the applicability of the non-probabilistic concepts from an objective viewpoint. Therefore, for each concept, its compatibility with the definitions of uncertainty and variability in the previous section is discussed. This discussion focusses on the ability to objectively represent the available information. In order to enable a critical review of the capabilities of the non-probabilistic concepts, this section first starts with a brief discussion on the main features of the probabilistic concept in the framework of uncertainty and variability modelling.

2.2.1 Basic properties of the probabilistic concept

In the classical frequentist application of the probabilistic concept, the goal of a numerical property description is to define a domain of possible values this property can adopt, and to give information on the frequency of occurrence of the numerical values in this domain. This is typically done by defining a probability density function $f_X(x)$ (PDF) for the probabilistic quantity X over the domain of possible values. The probability that the quantity lies within the interval $[a,b]$ is directly derived from this PDF:

$$P(a \leq X \leq b) = \int_a^b f_X\left(x\right) dx \tag{1}$$

The expectation of a function $g(X)$ with respect to $f_X(x)$ is defined as:

$$E\{g(X)\} = \int_{-\infty}^{\infty} g(x) f_X(x) dx \tag{2}$$

The mean value or average of the distribution $f_X(x)$ equals $E\{X\}$. The other most dominant features of a probabilistic quantity are commonly described by the central moments associated with the PDF. The n^{th} central moment m_n follows from the mean value using:

$$m_n = \int_{-\infty}^{\infty} (x - E\{X\})^n \, f_X\,(x)\, dx \tag{3}$$

The second order central moment is the most commonly used, and referred to as the variance of the distribution denoted by $var(X)$. The standard deviation is defined as $\sigma_X = \sqrt{var(X)}$. It is a common measure for the dispersion of the distribution about the mean value.

For multiple probabilistic quantities, the PDF concept is extended to more dimensions using the joint probability density function $f_{X_1 \ldots X_n}(x_1, \ldots x_n)$. The expectation is defined in an analogous manner as for the univariate distributions. The covariance, the first order joint central moment, gives a measure of the interdependence between the quantities. It is commonly represented by the covariance matrix Γ containing all individual variances and covariances:

$$\Gamma = \begin{bmatrix} var(X_1) & cov(X_1, X_2) & \ldots & cov(X_1, X_n) \\ cov(X_2, X_1) & var(X_2) & \ldots & cov(X_2, X_n) \\ \vdots & \vdots & \vdots & \vdots \\ cov(X_n, X_1) & cov(X_n, X_2) & \ldots & var(X_n) \end{bmatrix} \tag{4}$$

with

$$cov(X_i, X_j) = E\{(X_i - m_{X_i})(X_j - m_{X_j})\} \tag{5}$$

Extensive literature exists on the subject of probability theory, treating a vast variety of PDFs and their applicability for description of random quantities. An overview of these can be found in [17] and [18].

In most available non-deterministic FEM software codes, the probabilistic concept is applied to describe both variabilities and uncertainties in a model. This is mainly due to the fact that there exists a large number of numerical analysis procedures based exclusively on probabilistic input quantities. Therefore, every non-deterministic quantity in a model is readily replaced by a probabilistic quantity by introducing an appropriate PDF. However, the probabilistic model does not necessarily represent the available objective information. For the study of the applicability of the probabilistic model, distinction between certain variabilities, uncertain variabilities and invariable uncertainties is necessary.

It is clear that the probabilistic concept is most appropriate to represent *certain variabilities*, since in the frequentist interpretation, the probabilistic description using a PDF is completely consistent with the definition of a variability as in section 2.1.1. The information on the range and the likelihood of a certain variability can be unambiguously incorporated in the PDF. Furthermore, the probabilistic outcome of the analysis will give an indication of the actual expected frequency of occurrence of the analysed phenomenon. It is, however, important that all information is available in order for the model to realistically represent the variability. For instance, if more than one variable property is present in the model, the correlation between the different variabilities might play an important role in the probabilistic analysis. Ideally, the joint PDF describing the likelihood and interdependence of all non-deterministic model properties is available. Since this is almost never the case, the probabilistic description of variability interdependence is generally limited to some moments of low order. Often, when cross correlations are unknown, the variabilities are assumed to be independent of one another.

For *uncertain variabilities*, a representation by a single random quantity is generally not sufficient. Engineering scientist FREUDENTHAL [19] who was one of the pioneers of probabilistic methods in engineering states that " *. . . ignorance of the cause of variation does not make such variation random.*". By this, he means that when crucial information on a variability is missing, it is not good practice to model it as a probabilistic quantity represented by a single random PDF. On the contrary, in this case it is mandatory to apply a number of different probabilistic models to examine the effect of the chosen PDF on the result. For instance, when the range of the variability is known but the information on the likelihood is missing, all possible PDFs over the range should be taken into consideration in the analysis. The analyst will generally select only a few probabilistic models which he considers consistent with the limited available information or most appropriate to obtain as much knowledge as possible on the result.

Most often, *invariable uncertainties* are represented by random quantities in probabilistic analysis. As such, the analyst tries to express his lack of knowledge of the property. This means that some PDF is chosen which to the knowledge of the analyst represents best the uncertain nature of the quantity, but which is not based on available objective information. It is clear that in this case, the information contained in the random quantity does not represent the actual variation of the quantity in the final product, since by definition, the invariable uncertainties are considered to be constant. The random quantity in this case merely represents the presumed likelihood that a model parameter will adopt a value. As such, the lack of knowledge is filled by subjective information provided by the analyst, expressed in the form of a PDF. This is sometimes referred to as a subjective PDF. In this context, Bayesian methods are becoming increasingly popular for the modelling of subjective uncertainty. The main advantage of using the probabilistic approach for subjective uncertainty modelling is that the available probabilistic procedures can be readily applied for the analysis. It should be kept in mind, however, that the main strength of the Bayesian approach is its capability of incorporating objective information that becomes gradually available. When this is not the case, the Bayesian approach remains a fully subjective representation of reality.

At this point, it is very important to emphasize the consequences of the difference in the use of the probabilistic concept for variabilities on the one hand, and invariable uncertainties on the other hand. The former represents variability defined as a variation from unit to unit or in time for the final product, while the latter clearly may not be interpreted in this sense. Consequently, when interpreting the results of a probabilistic analysis based on both uncertainties and variabilities, it is imperative to distinguish between the different meanings attached to both. Though this seems straightforward, neglecting this distinction is a very common mistake in probabilistic uncertainty analysis. Section 3.3.2 elaborates further on the implications of this problem for probabilistic reliability analysis.

Recently, some criticism on the general application of probabilistic methods is arising. A first argument concerns the necessity of the probabilistic analysis. In some cases, non-deterministic analysis is merely a tool to enhance or optimise the expected physical behaviour of a design based on (limited) knowledge of external non-deterministic influences. While probabilistic analyses are applicable for this purpose, probabilistic information on the behaviour of a design is not always primordial. Especially when subjective information is present in the analysis, other non-probabilistic techniques could give a valuable, maybe even additional insight into the non-deterministic nature of the simulated behaviour. Whether or not these techniques are valuable alternatives depends on the added value of the results. A second often heard argument against probabilistic analysis relates to its computational time-efficiency. This refers to the popular implementation using a Monte Carlo simulation, which is a rather time consuming technique, as it uses a high number of deterministic calculations to simulate the probabilistic process. Therefore, its computational efficiency will always lag behind the efficiency of the corresponding deterministic analysis.

Combining both arguments above, the criticism comes down to the fact that for some cases, the added value of the results of a probabilistic analysis does not justify the computational effort required to obtain them. In order to objectively assess this criticism as an argumentation in favour of non-probabilistic concepts, a clear insight in the available alternative techniques is necessary. Therefore, the next sections describe the basic properties of two non-probabilistic concepts, after which section 3 will discuss their applicability for engineering design purposes.

2.2.2 Basic properties of the interval concept

Recent developments in interval arithmetics are mainly based on the work of MOORE [20], who introduced interval vectors and matrices and the first non-trivial applications. By definition, an interval scalar consists of a single continuous domain in the domain of real numbers $\mathbb{R}$. The range is bounded by a lower and an upper bound. The domain of interval scalars defined over $\mathbb{R}$ is denoted by $\mathbb{IR}$. In order to facilitate the development of the mathematical background in the remainder of this work, this section first introduces a generalised notation for intervals and sets.

A general *interval scalar* is denoted by a boldface variable $\mathbf{x}$. The lower and upper bound of an interval scalar $\mathbf{x}$ are denoted by $\underline{x}$ respectively $\overline{x}$. A real closed interval scalar is defined as:

$$\mathbf{x} = \left\{ x \mid (x \in \mathbb{R}) \left(\underline{x} \leq x \leq \overline{x} \right) \right\} \tag{6}$$

An alternative notation for an interval $\mathbf{x}$ is $[\underline{x}, \overline{x}]$. A straightforward extension of an interval scalar is a general *set scalar* denoted by $\langle x \rangle$. It consists of a number of disjoint interval scalars in $\mathbb{IR}$:

$$\langle x \rangle = \bigcup_{i=1 \ldots n} \mathbf{x}_i \tag{7}$$

Combining different interval scalars into a vector or matrix is generally done by simply combining all component intervals independently in the multidimensional space. According to this principle, an interval matrix or vector includes any combination of the entries as long as they are within the bounds of their interval scalar. This means that all entries are implicitly assumed to be mutually independent quantities. This has very important consequences for the use of the interval concept in FE analysis since there is generally a strong dependency between FE vector or matrix entries. Neglecting this dependency results in the implicit introduction of conservatism into the analysis.

Mathematically, the *interval matrix* $[\mathbf{X}] \in \mathbb{IR}^{n \times m}$ describes the set of all matrices for which each matrix entry x_{ij} is contained within its corresponding interval scalar $\mathbf{x_{ij}}$:

$$[\mathbf{X}] = \Big\{ [X] \mid x_{ij} \in \mathbf{x_{ij}} \Big\} \tag{8}$$

An *interval vector* similarly is denoted by $\{\mathbf{x}\} \in \mathbb{IR}^n$. Finally, a *set matrix* $\big\langle [X] \big\rangle$ describes the set of all possible matrices where each matrix entry x_{ij} is contained within its corresponding set scalar $\langle x_{ij} \rangle$:

$$\big\langle [X] \big\rangle = \Big\{ [X] \mid x_{ij} \in \langle x_{ij} \rangle \Big\} \tag{9}$$

Again, the matrix elements are implicitly considered independent.

The information represented by an interval object depends on the type of modelled non-deterministic quantity. Also here, distinction between certain variabilities, uncertain variabilities and invariable uncertainties is necessary.

For *certain variabilities*, the input interval objects are derived from the support of the corresponding input PDFs. Consequently, the result of an interval analysis only represents the actual range of the variable outcome of the analysis. The available information on the likelihood inside the range is lost, which is an important disadvantage. Especially for a variability with a justifiable PDF support that is very large, using the support as input for the interval analysis will generally result in an extremely wide output interval. While it is theoretically correct to state that the final result will range over this output interval, disregarding the probability of the PDF tails in this case clearly strongly devaluates the interval analysis.

When the upper and lower bounds of a non-deterministic property are well-defined but information on the type of the distribution is missing, it belongs to the class of *uncertain variabilities*. In this case, the interval model represents perfectly the available information. However, especially for variabilities with a very large PDF support, the determination of the corresponding interval bounds is not always unambiguous, since the probability of the values that are located in the tails of the commonly applied PDFs with large support is typically very low. If these tails cannot be justified adequately with experimental data, there is no reason to unconditionally use the PDF support for the interval analysis. In this case, the analyst should implement the bounds which he considers realistic with respect to the available experimental data. Often, the 3σ-bounds are assumed to be realistic interval bounds. This conversion does not necessarily reduce the truthfulness of the uncertainty representation when there is little information on the actual tails of the PDF. Still, if the tails of the PDF are expected to have little probability, the impact of the subjective interval bounds on the interval analysis result is much larger than the impact of subjective PDF support limits on the probabilistic analysis result. Therefore, variabilities with unknown PDF support but a well-known normal-like behaviour near the center of the PDF are best modelled probabilistically.

For *invariable uncertainties*, generally a subjective interval is required. In this case, care should be taken not to interpret the interval quantity as the actual range in the physical product. It merely represents the values the analyst considers possible at the time the analysis is performed. Therefore, similar to the application of the probabilistic concept for invariable uncertainties, it is important to acknowledge the subjectivity in the result of the analysis. However, since the interval concept requires less subjective information to be added to the problem description, there is less room for misinterpretation of the results.

To conclude, we can state that the probabilistic concept remains the most valuable for the representation of certain variabilities and uncertain variabilities with unknown support but known normal-like behaviour. The omission of a known PDF through the interval concept can only be justifiable when probabilistic information is not required, or the computational cost of the interval analysis is significantly lower. The interval concept is most valuable when dealing with uncertain variabilities with known support but unknown distribution, or invariable uncertainties.

2.2.3 Basic properties of the fuzzy concept

The theory of fuzzy logic was introduced by ZADEH [21] in 1965, and has gained an increasing popularity during the last two decades. Its most important property is that it is capable of describing linguistic and, therefore, incomplete information in a non-probabilistic manner.

A fuzzy set can be interpreted as an extension of a classical set. Where a classical set clearly distinguishes between members and non-members of the set, the fuzzy set introduces a degree of membership, represented by the *membership function*. This membership function describes the grade of membership to the fuzzy set for each element in the domain. The concept allows membership values different from zero and one. This enables the representation of a value that is only to a certain degree member of the set.

For a fuzzy set $\tilde{x}$, the membership function is defined as $\mu_{\tilde{x}}(x)$ for all x that belong to the domain X:

$$\tilde{x} = \Big\{ \big(x, \mu_{\tilde{x}}(x)\big) \mid \big(x \in X\big)\big(\mu_{\tilde{x}}(x) \in [0,1]\big) \Big\} \tag{10}$$

If $\mu_{\tilde{x}}(x) = 1$, x is definitely a member of the subset $\tilde{x}$. If $\mu_{\tilde{x}}(x) = 0$, x is definitely not a member of the subset $\tilde{x}$. For every x with $0 < \mu_{\tilde{x}}(x) < 1$, the membership is not certain. For the representation of numerical uncertainty, a class called *normal fuzzy numbers* is generally used. For these fuzzy numbers, there is at least one point where the membership is equal to one, and the membership is strictly increasing and decreasing to the left respectively the right of this point. The most frequently applied shapes for the membership functions are the triangular and Gaussian shape. Figure 3 shows the membership functions of some typical normal fuzzy numbers.

While the concept of fuzzy logic was invented in 1967, it resulted mainly in practical applications during the last two decades. The works of DUBOIS and PRADE [22, 23] contributed to a large extent to this evolution. The concept has been most successful in the application to controller design, known as *fuzzy control* [24]. In a fuzzy controller, the fuzzy concept is the basis for a human-like decision process for choosing an appropriate process control value based on fuzzily described state variables. This human-like concept does not require strict mathematical control rules, but is capable of handling linguistic rules often based on expert knowledge rather than strict objective data.

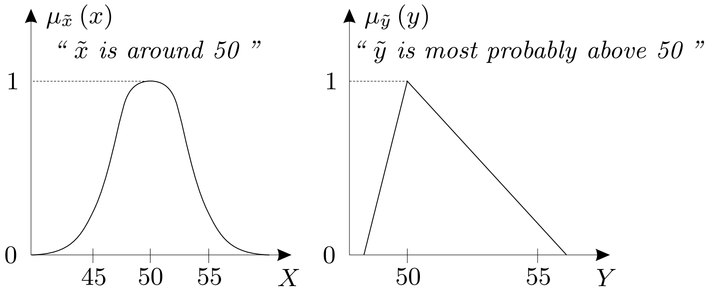

Figure 3: Some typical membership functions that describe linguistic variables

ZADEH [25] extended the theory of fuzzy sets to a basis for reasoning with possibility. In this interpretation, the membership function is considered as a possibility distribution function, providing information on the values that the described quantity can adopt. More generally, the possibility is defined as *a subjective measure that expresses the degree to which the analyst considers that an event can occur*. It provides in a system of defining intermediate possibilities between strictly impossible and strictly possible events. Through this interpretation, the fuzzy concept has become a tool to model subjective knowledge numerically in a non-probabilistic concept. This has drawn the attention of the numerical community, since knowledge of uncertainties in a numerical model is commonly based on expert opinion. This has lead to the first attempts to use the fuzzy concept in a non-deterministic framework, resulting in some applications in structural optimisation under uncertainty [26, 27]. Also, this has initiated the development of the Fuzzy Finite Element Method (FFEM) for numerical analysis of non-deterministic models [28].

However, the application of the fuzzy concept for non-deterministic numerical modelling is not straightforward. The main problem of the representation of a model property through a fuzzy set, is that the membership function does not relate to an objective measurable quantity. The level of membership that is assigned to different members of a fuzzy set is completely based on the subjective beliefs of the analyst. Therefore, also the fuzzy results obtained from the analysis will be biased with the subjective input. Hence, these results may only be interpreted in reference to the assumed fuzzy input. This poses an important restriction on the use of the fuzzy approach for numerical design validation purposes. The practical consequences of this restriction will be further addressed in section 3.3. First, the applicability of the fuzzy concept for variability and uncertainty representation is discussed.

For a fuzzy representation of *certain variabilities*, the known PDF has to be converted to a compatible membership function. A number of methods have been developed for this purpose [23, 29]. The basic law for the conversion follows from the consistency principle, which states that the degree of possibility of an event is greater than or equal to its degree of probability. This principle implies the following rule for conversion of probabilistic into possibilistic distributions:

$$\int_B f_X(x)\,dx \;\leq\; \max_{x\in B}\left(\frac{\mu_{\tilde{x}}(x)}{\max \mu_{\tilde{x}}(x)}\right) \tag{11}$$

for any set B in the feasible domain. This means that even a completely known probabilistic quantity has an infinite number of possibilistic representations. Therefore, these conversion techniques always rely on some sort of subjective judgement. Still, it is the author's opinion that forcing the application of fuzzy sets into the domain of certain variabilities through a conversion of PDFs as described above is rather irrational. Available objective probabilistic data is replaced by a subjective description, resulting in the loss of very valuable information. This loss is generally unjustifiable. Therefore, the conversion of a PDF to a membership function should not be done.

For *uncertain variabilities*, the fuzzy concept can be used for a hybrid uncertainty model. It stems from an alternative interpretation of a possibility distribution introduced by DUBOIS and PRADE [30] based on the *Evidence Theory* [31]. In this approach, a fuzzy number is used to represent a class of probability random quantities that have a cumulative distribution function (CDF) in between boundaries derived directly from the possibility distribution. The left boundary on the compatible CDFs coincides with the increasing branch of the fuzzy number. The right boundary coincides with the complement of the decreasing branch of the fuzzy number. Figure 4 clarifies this approach. In this concept, the possibilistic approach becomes a tool to simultaneously examine the effect of a set of different PDFs in a single analysis. While the ability of this concept to model classes of probabilistic data seems extremely powerful, it has only been applied very rarely in uncertainty analysis.

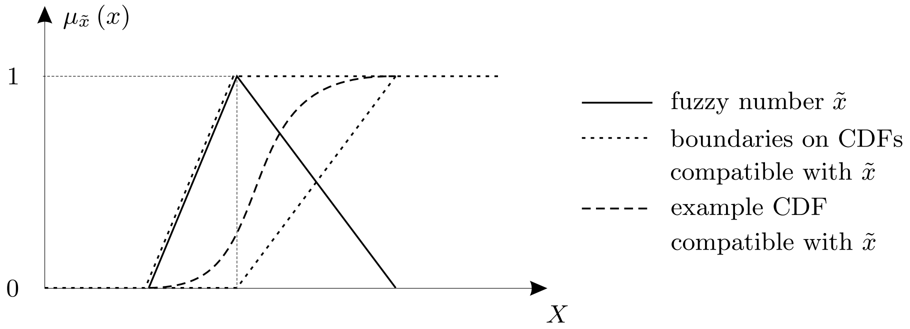

Figure 4: Possibility distribution of a fuzzy number and corresponding lower and upper boundaries for CDF compatible with the fuzzy number

Finally, an *invariable uncertainty* requires a fuzzy set that represents the subjective expectation of the analyst. When the invariable uncertainty represents an open design decision subject to optimisation, the analyst can express his preference of the quantity through the possibility distribution. Still, when interpreting the results, reference to the chosen input membership functions is imperative.

Considering the explicit subjective nature of a fuzzy set, it is concluded that it is most useful to describe uncertainties. The more objective information becomes available on a non-deterministic model property, the less the fuzzy concept is appropriate to describe it.

3 Non-deterministic FE analysis for design purposes

The previous section clearly shows that the interval and fuzzy approach can be very valuable concepts for modelling incomplete information under specific circumstances. It is clear that the capability of modelling different amounts of available information creates new possibilities for design engineers. This section will give an overview of these possibilities in the context of the finite element analysis. In sections 3.1 and 3.2, the translation of the classical deterministic FE analysis procedure to respectively the interval and the fuzzy approach is described. The knowledge of the underlying numerical procedures is primordial for a good understanding of the discussion on the practical application for design purposes in section 3.3.

3.1 The interval FE analysis

The goal of the IFE analysis is to obtain the maximal meaningful information on the possible outcome of the FE analysis when each model parameter is expressed either by an interval or by a crisp number. Numerically, this is equivalent to finding the minimal and maximal deterministic analysis results taking into account all possible models that lie within the interval uncertainty description. In this section, we consider the FE analysis in a black-box form, i.e. , a mapping of input properties contained in the FE model to output quantities derived from the FE solution. The input of this function consists of all non-deterministic model properties, assembled in a parameter vector $\{x\}$. The output quantities can be any set of system response quantities derived from the FE analysis result, from nodal displacements over stresses to other derivative quantities. This mapping in its most simple form is represented by the black-box function $f\big(\{x\}\big)$ resulting in the output vector $\{y\}$. In the interval analysis, the input parameter vector is defined to be contained within an interval vector $\{\mathbf{x}\}$. The IFE procedure then is numerically equivalent to finding the following solution set:

$$\langle\{y\}\rangle = \left\{ \{y\} \mid \Big(\{x\} \in \{\mathbf{x}\}\Big)\Big(\{y\} = f\big(\{x\}\big)\Big) \right\} \tag{12}$$

Generally, the components of the output vector $\{y\}$ are related to each other through the design parameters. Therefore, the solution set $\langle\{y\}\rangle$ can basically adopt any form in the output space. This makes it extremely difficult to calculate an exact description of the solution set. In most cases, however, only the individual ranges of some components of the result vector are really of interest. Therefore, research focuses on calculating an interval vector that approximates the exact solution set but neglects the interdependencies between the output vector components. This is referred to as a hypercubic approximation of the result. It describes a range for each vector component, but not all combinations of vector components within these ranges are part of the exact solution set. Generally, the smallest hypercube around the exact solution set is of most importance. Still, in early design stages, also conservative hypercubic approximations of the solution set can be very valuable. Figure 5 gives a two-dimensional illustration of an exact solution set and the corresponding approximate hypercubes.

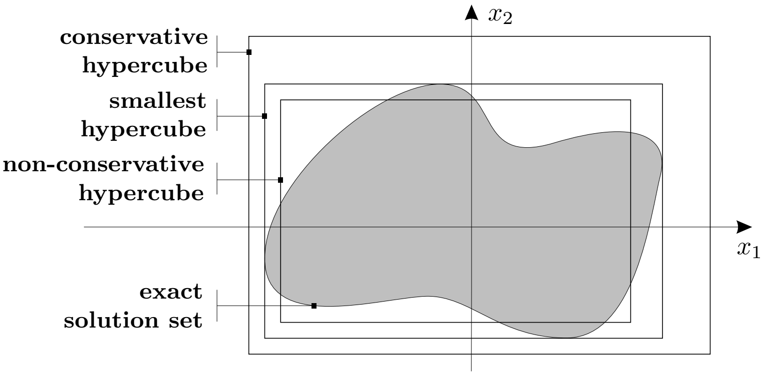

Figure 5: Hypercubic approximations of a continuous two-dimensional output set of an IFE analysis

Research nowadays in this area mainly focusses on three different numerical solution strategies to calculate a hypercubic approximation of the exact solution set: the vertex analysis, the global optimisation approach and the interval arithmetic approach. All will be briefly described below. Recently, the current authors have proposed a hybrid approach which will also be briefly discussed here. A more detailed overview of implementations of the IFE methodology based on these principles can be found in [32].

3.1.1 The vertex analysis

DONG et al. [33] first introduced the vertex method. This method approximates the range of the result of a numerical procedure by introducing all possible combinations of the boundary values of the input intervals into the analysis. For N input intervals, there are 2^N vertices for which the analysis has to be performed. These vertices are denoted by $\{c_j\}, j = 1, \dots 2^N$. Each of these represents one unique combination of lower and upper bounds on the N input intervals. The approximate analysis range is deduced from the extreme values of the set of results for these vertices:

$$\{\mathbf{y}\} \approx \left[\min_j f\big(\{c_j\}\big), \max_j f\big(\{c_j\}\big) \right] \tag{13}$$

Despite its simplicity, this method has some important disadvantages. It is clear from equation (13) that the computational cost increases exponentially with the number of input intervals. This limits the applicability of the vertex method to rather small systems, or systems with very few interval entries in the system matrices. The main disadvantage of this method, however, is that it cannot identify local optima of the analysis function which are not on the vertex of the input space. It only results in the smallest hypercube if the analysis function is monotonic over the considered input range. This is a strong condition that is difficult to verify for FE analysis because of the complicated relation of analysis output to physical input uncertainties. The approximation obtained when monotonicity is not guaranteed is not necessarily conservative. This fact reduces the validity of this method for design validation purposes.

3.1.2 The global optimisation strategy

In essence, calculating the smallest hypercube around the solution set expressed in equation (12) is equivalent to performing a global optimisation, aimed at the minimisation and maximisation of the components of the deterministic analysis results $\{y\}$. The deterministic FE analysis is the goal function of the optimisation and the uncertain parameters are the design variables. The interval vector in which the uncertain parameters are contained defines the constraints for the variables. The optimisation is performed independently on every element of the result vector $\{y\}$. Therefore, the solution set of equation (12) becomes an interval vector describing the hypercube around the exact solution:

$$\{\mathbf{y}\} = \begin{Bmatrix} \mathbf{y_1} \\ \mathbf{y_2} \\ \vdots \\ \mathbf{y_n} \end{Bmatrix} \tag{14}$$

with:

$$\begin{aligned} \underline{y}_i &= \min_{\{x\}\in\{\mathbf{x}\}} f_i\big(\{x\}\big),\ i = 1\ldots n && (15) \\ \overline{y}_i &= \max_{\{x\}\in\{\mathbf{x}\}} f_i\big(\{x\}\big),\ i = 1\ldots n && (16) \end{aligned}$$

An efficient and robust optimisation algorithm is primordial for this solution strategy. RAO et al. [4] applied POWELL's method to tackle the optimisation. KÖYLÜOĞLU [1] defined a linear programming solution for this purpose. The input interval vector defines the number of constraints and, therefore, strongly influences the performance of the procedure. Also, because of the required execution of the deterministic FE analysis in each goal function evaluation, the optimisation approach is numerically expensive. Therefore, this approach is best suited for rather small FE models with a limited number of input uncertainties.

3.1.3 The interval arithmetic strategy

The interval arithmetic approach consists of translating the complete deterministic numerical FE procedure to an interval procedure using the arithmetic operations for addition, subtraction, multiplication and division of interval scalars:

$$\begin{aligned} \mathbf{a}+\mathbf{b} &= \big[\underline{a}+\underline{b}, \overline{a}+\overline{b}\big] && (17) \\ \mathbf{a}-\mathbf{b} &= \big[\underline{a}-\overline{b}, \overline{a}-\underline{b}\big] && (18) \\ \mathbf{a}\times\mathbf{b} &= [\min(\underline{a}\times\underline{b}, \underline{a}\times\overline{b}, \overline{a}\times\underline{b}, \overline{a}\times\overline{b}) \quad \max(\underline{a}\times\underline{b}, \underline{a}\times\overline{b}, \overline{a}\times\underline{b}, \overline{a}\times\overline{b})] && (19) \\ \mathbf{a}/\mathbf{b} &= \mathbf{a}\times\left[\tfrac{1}{\overline{b}}, \tfrac{1}{\underline{b}}\right],\ \text{ if } 0\notin\mathbf{b} && (20) \end{aligned}$$

The outline of the interval procedure corresponds completely to the deterministic FE analysis. The main difference is that each substep of the interval algorithm calculates the range of the intermediate subfunction instead of the deterministic result. Application of this principle on the complete algorithm results in the range of the output of the analysis.

There is an important drawback for this method. The inclusion property for ranges of nested functions states that an arithmetic interval operation introduces conservatism in its result if it neglects correlation that exists between the operands. A simple example illustrates this. Consider the function:

$$f\left(x\right)=x^{2}-x \tag{21}$$

applied on the interval $\mathbf{x}=[0,1]$. Applying interval arithmetic, both terms are considered independently. This results in the interval solution $[-1,1]$. However, the exact range of the function equals $\left[-\frac{1}{4},0\right]$.

In an automatic computer procedure as for the IFE analysis, this phenomenon cannot be avoided because it is impossible to keep track of the relationships between all intermediate results of the algorithm. Consequently, each interval substep results in an enclosure of the exact substep range. Therefore, also the final result is a conservative approximation of the exact range of the FE analysis. Generally, the degree of conservatism is unknown. It possibly can be too high to be useful for practical applications.

The implementation of the interval arithmetic IFE approach consists of two parts:

1. The translation of the input intervals to an interval system description in the form of interval system matrices. These are obtained by translating the deterministic assembly procedure to interval analysis and results in the interval system matrices.

2. The approximation of the solution of the analysis expressed as an interval problem using the interval system matrices. For structural dynamic analysis, these are denoted by $[\mathbf{K}]$, $[\mathbf{M}]$ and $[\mathbf{C}]$. The exact solution set then becomes:

$$\left\{\{y\}\ \middle|\ \Big([K]\in[\mathbf{K}]\Big)([M]\in[\mathbf{M}]\Big)([C]\in[\mathbf{C}]\Big)\Big(\{y\}=f\big([K],[M],[C]\big)\Big)\right\} \tag{22}$$

   with f the function representing the calculation of the analysis result based on the system matrices.

While nearly all literature on IFE is based on the solution phase, the interval matrix assembly phase was shown to have a very important contribution to the conservatism in the final analysis results [32]. Still, the applicability of the interval arithmetic strategy mainly depends on the availability of a calculation procedure for the hypercubic approximation of the solution set of equation (22). This procedure strongly depends on the numerical properties of the intended FE analysis.

For static analysis, the second phase of the interval analysis consists of an equilibrium or steady-state problem. This is numerically equivalent to a matrix equation that requires the solution of a system of equations. The corresponding interval problem yields:

$$\langle\{y\}\rangle=\left\{\{y\}\ \middle|\ \Big([A]\in[\mathbf{A}]\Big)\Big(\{b\}\in\{\mathbf{b}\}\Big)\Big([A]\{y\}=\{b\}\Big)\right\} \tag{23}$$

with $\{\mathbf{b}\}$ the interval vector representing the uncertain generalised loading of the considered model. This solution set contains all vectors $\{y\}$ which are a solution of the matrix equation $[A]\{y\}=\{b\}$ with $[A]$ and $\{b\}$ ranging respectively over the interval objects $[\mathbf{A}]$ and $\{\mathbf{b}\}$. From the world of interval arithmetic, this solution set is referred to as the *united solution set* [34] and denoted by $\Sigma_{\exists\exists}\big([\mathbf{A}],\{\mathbf{b}\}\big)$. There have been many attempts in literature to calculate this solution set. However, up to now, these attempts are not really successful. Most algorithms are either limited to very small scale systems, or can only be applied under very restrictive conditions.

For dynamic analysis, the situation is different. Especially for interval arithmetic implementations of the eigenvalue problem, some successful algorithms have been defined. For this application, the IFE procedure requires the calculation of the solution set:

$$\left\{\lambda_i\ \middle|\ \Big([K]\in[\mathbf{K}]\Big)\Big([M]\in[\mathbf{M}]\Big)\Big([K]\{\phi_i\}=\lambda_i[M]\{\phi_i\}\Big)\right\} \tag{24}$$

with $[K]$ and $[M]$ incorporating implicitly the dependency of the system matrices on the input parameters. It can be shown that, assuming independent interval system matrices, the bounds of this exact solution set are achieved for vertex matrix combinations [35]. This means that the exact solution of the interval eigenvalue problem can be found. Some algorithms have been developed which efficiently calculate this exact vertex solution. CHEN et al. [35] introduced a non-iterative procedure based on the RAYLEIGH quotient, which states that the lower and upper bound on an eigenvalue follow directly from two deterministic eigenvalue problems. An enhanced methodology was developed by EL-GEBEILY et al. [36]. It provides a solution for the original problem with an extra restriction of symmetry on the considered system matrices:

$$\left\{\lambda_i \mid \Big([K_s] \in [\mathbf{K}]\Big)\Big([M_s] \in [\mathbf{M}]\Big)\Big([K_s]\{\phi_i\} = \lambda_i\,[M_s]\{\phi_i\}\Big)\right\} \tag{25}$$

with K_s and M_s symmetric. The most important effect of this extra restriction is that it intrinsically removes the conservatism resulting from allowing artificial non-symmetric system matrices. The numerical procedure is based on the interval translation of the deterministic STURM sequence. It proves to be an efficient iterative algorithm. Unfortunately, it is limited to tridiagonal system matrices. This makes it only applicable for specific cases.

3.1.4 Hybrid IFE analysis

In order to extend the applicability of IFEM, a general remedy to excessive conservatism was introduced [37]. It is a hybrid procedure, consisting of both a global optimisation and an interval arithmetic part. In the first part, an optimisation is applied to calculate the interval result at some intermediate step of the total algorithm. In the second part, the interval analysis is performed on these intermediate results. This method has two major advantages:

- because of the global optimisation, all conservatism prior to the optimised intermediate result is neutralised
- the performance of the optimisation step is controllable by adequately choosing the level on which to perform it

This approach has been successfully applied in an IFE procedure for the calculation of interval FRFs [28]. In the first part of this procedure, the optimisation is used to translate the interval properties defined on the FE model to the exact interval modal stiffness and mass parameters of the structure. The calculation of the envelope FRFs in the second part is done by applying the interval arithmetic equivalent of the modal superposition procedure on these interval modal parameters. This procedure neutralises all conservatism in the matrix assembly phase, since it directly uses the modal parameters as goal functions in the optimisation part. The final envelope FRFs have been proven to contain only a very limited amount of conservatism.

3.2 The fuzzy FE analysis

The principal goal of the FFE analysis is to obtain the membership function of the output quantities given the membership functions of all input quantities. It basically requires a concept to handle the combination of the fuzzy input sets, i.e. , a definition of a Cartesian product combining different fuzzy sets. The membership function of a Cartesian product of variables described by individual membership functions has been defined in literature [38]:

$$\mu_{\tilde{x}_1 \times \ldots \times \tilde{x}_n}(x_1, \ldots x_n) = \min\Big(\mu_{\tilde{x}_1}(x_1), \ldots \mu_{\tilde{x}_n}(x_n)\Big) \tag{26}$$

This definition states that the possibility of a combination of fuzzy events equals the minimum of the possibilities of all individual events.

Calculating the result of the FFE analysis further requires an arithmetic that handles the numerical evaluation of functions of fuzzy sets. A general concept follows directly from ZADEH's extension principle [39], which describes a general procedure for extending crisp mathematical procedures to fuzzy quantities. It states that the fuzzy output $\tilde{y}$ of the crisp function $f(x_1, x_2, \ldots, x_n)$ applied to n fuzzy numbers $\tilde{x}_i$ equals:

$$\begin{cases} \mu_{\tilde{y}}(y) = \sup\limits_{\substack{x_1,\ldots x_n \\ y=f(x_1,\ldots x_n)}} \Big(\min\big(\mu_{\tilde{x}_1}(x_1), \ldots \mu_{\tilde{x}_n}(x_n)\big)\Big) \\ \\ \mu_{\tilde{y}}(y) = 0 \quad \text{if} \quad f^{-1}(y) = \emptyset \end{cases} \tag{27}$$

This definition implies that the membership value of the fuzzy result $\tilde{y}$ for a specific value y^* equals the largest among the membership values $\mu_{\tilde{x}_1 \times \ldots \tilde{x}_n}$ of all input combinations $(x_1, \ldots x_n)$ resulting in y^*. The input combinations which result in y^* are referred to as realisations of y^*. The possibilistic interpretation of the extension principle is that if a value y^* can be achieved for different combinations of the input quantities, it will adopt its degree of possibility from the realisation with the highest degree of possibility.

The definition of the extension principle as in equation (27) is not readily implementable. For each value y of the observed output domain, it requires the complete set of realisations in the input domain to derive the membership value. This is an extremely difficult task. An alternative approach consists of searching in the output domain for sets that have an equal degree of membership. This is achieved by analysing the input domain on a specific level of membership α. At this level, the α-cuts of the input quantities are defined as:

$$\mathbf{x_i}_\alpha = \big\{x_i \in X_i, \mu_{\tilde{x}_i}(x_i) \geq \alpha\big\} \tag{28}$$

This means that an α-cut is the interval resulting from intersecting the membership function at $\mu_{\tilde{x}_i}(x_i) = \alpha$ (see figure 6). After deriving the α-cuts of all input quantities at a specific level, a general interval analysis is performed on these intervals:

$$\mathbf{y}_\alpha = \Big\{y \mid \Big(x_i \in \mathbf{x_i}_\alpha, \forall i\Big)\Big(y = f\big(\{x\}\big)\Big)\Big\} \tag{29}$$

There are two important observations concerning the resulting output interval of this analysis:

- The output interval results from combining all possible input quantities with a membership value larger than or equal to α. Therefore, following the extension principle, all output values in the interval have a membership value which equals at least α.
- Values outside this interval cannot be obtained using exclusively input values with membership values larger than or equal to α. Therefore, all values outside the interval have a membership smaller than α.

These observations basically state that the obtained output interval is an intersection of the output membership function at the α-level, and consequently represents an α-cut of the output. This means that a discretised approximation of the output membership function can be obtained from repeating the α-level procedure at a number of levels. Figure 6 clarifies this procedure for the FFE analysis. As such, the IFE analysis has become the numerical core of the FFE analysis. Therefore, all implementation principles for the IFE method as discussed in the previous section can be easily extended to a FFE procedure using the α-cut strategy.

3.3 Application of non-probabilistic approaches in a design process

Using the above implementation techniques, virtually any FE analysis can be transformed into a non-probabilistic equivalent whenever the non-determinism in the model is best described in either the interval or fuzzy concept. Still, the decision on which non-deterministic concept to use should not be based exclusively on the available information at the analysis input. As shown in the example in section 2.1.3,

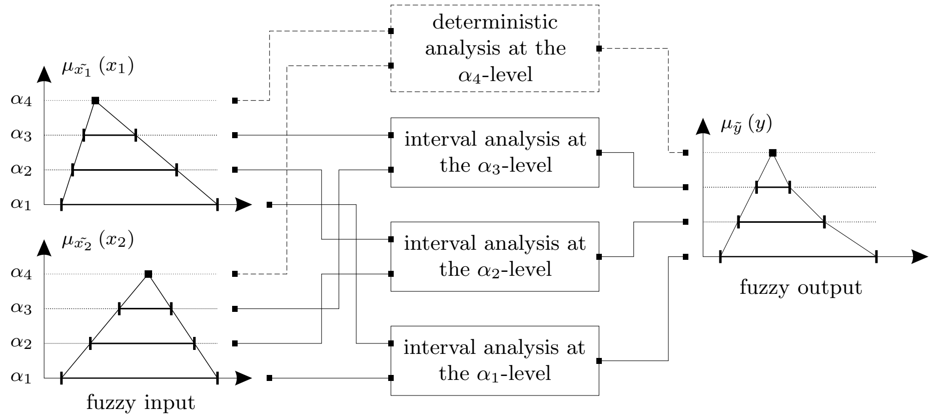

Figure 6: Scheme of the numerical procedure to perform a fuzzy FE analysis using 4 α-sublevels

the clear definition of the objective of the analysis is at least equally important in the determination of the most appropriate non-deterministic analysis tool. Therefore, this section focusses on a number of practical non-deterministic analysis types that concern a design engineer. Again, in order to evaluate the possibilities of the non-probabilistic approaches in specific applications, references will be made to the corresponding probabilistic treatment of the non-determinism.

3.3.1 Numerical non-determinism in a design process

The general objective of the application of numerical tools in a design process is to increase the design quality of an initial prototype by simulation of its realistic physical behaviour. While numerical analyses have proven to be very valuable in increasing design quality, an exact quantification of the design quality reached through the analysis is not always straightforward. This is mainly due to the fact that in any step of the design, there remains non-determinism in the numerical analysis. There generally is an evolution of the type of non-determinism encountered during a typical design process. Or as formulated by ROSS [40]: *As more information about a problem becomes available, the mathematical description of non-determinism can transform from one theory to the next in the characterization of the uncertainty as the uncertainty diminishes or, alternatively, as the information granularity increases and becomes specific.*

In an early stage, objective information on model properties is often difficult to obtain, since a large number of model properties have yet to be defined. Some design decisions are even intentionally postponed in order to be able to study their effect on the design quality. Furthermore, early design improvements are commonly the result of expert knowledge rather than detailed numerical procedures. This means that the amount of objective information on average is low, and therefore subjectiveness is substantially present in the analysis. This leads to the conclusion that in early design stages, most non-determinism should be classified as uncertainty. Through the course of a design process, the amount of information generally increases. In some cases, the non-deterministic properties can be more objectively described, e.g. , when certain design aspects are fixed, or component manufacturers are chosen. The remaining non-determinism gradually can be classified mainly as model variability, i.e. , design independent variations in the product or its environment.

The evolution of non-determinism in a typical design process as described above is illustrated in figure 7. As shown in the figure, the numerical prediction of the actual design quality improves over the design process. In the early stages, the non-determinism in the numerically predicted design quality is mainly driven by model uncertainties, whereas in later stages, variability becomes more important. This figure also indicates the evolution of the numerical concepts that are most appropriate for the dominant class of the occurring non-determinism.

The evolution of a property from one class of non-determinism to another can be clarified using a simple example. Take for instance the design of a new car body. The start point of the structural design is generally

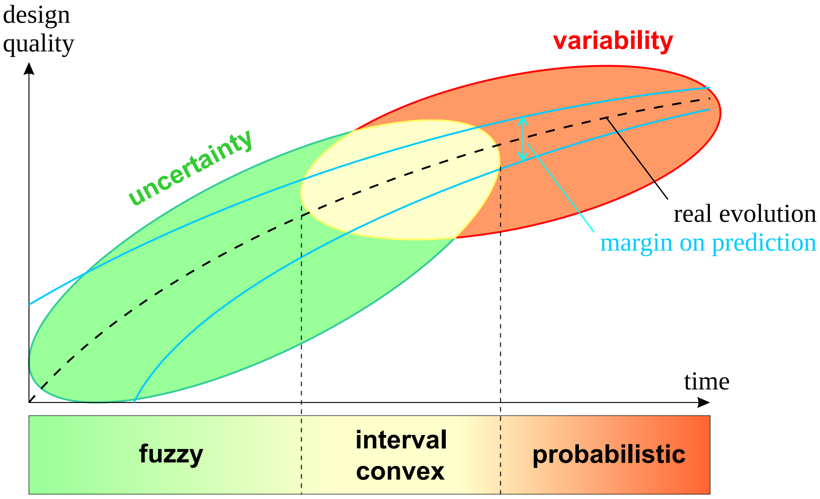

Figure 7: Typical occurrence of non-determinism in the product quality predictions during a design process

a conceptual design inspired esthetically rather than mechanically. In this initial design, there is a lot of non-determinism on the dimensions of structural components, such as for instance plate thickness. Since there is no information whatsoever on the exact plates that will be used, numerical analysis in this phase can only incorporate subjective knowledge based on other designs. Alternatively, a designer could be interested in the impact of a certain plate thickness on the behaviour of the design. In that case, a preferred range could be defined for the thickness in order to identify the most appropriate value. In either case, there is no clear objective information on the actual property in the final product. Hence, if non-determinism on this property is to be taken into account, this can only be achieved through modelling of subjective knowledge. Later, at a certain point in the design process, a specific reference value will be chosen for the thickness of the plates in the car body structure. Some tolerances are chosen, which define the allowable region for these properties in the actual product. At this point, the range of the thickness in the actual product is known, but there's no information on the likelihood inside the range. The property is clearly evolved to an interval. Finally, when the design is finalised up to the detailed description of the manufacturing process, information on the variation of the plate thickness within the bounds of the tolerances could become available. The value for the thickness then becomes a variability.

3.3.2 Numerical reliability analysis

The reliability of a product is defined as the likelihood that it will successfully fulfil its intended task over a predefined period in time under specific environmental conditions. Numerical reliability analysis based on probabilistic analysis is very popular because when realistic data is used, it can give a clear indication of the likelihood of failure of the analysed structure. As such, it can be usefully applied in an economical product analysis taking into account the cost associated with failure. This probabilistic reliability analysis is broadly applied and already incorporated in generally accepted design specifications in civil engineering. However, its application in mechanical engineering is far less standardised. This is mainly due to the plenitude of different mechanical products, which all require a different amount of reliability. Hence, there are very few standards for reliability in mechanical design. Each product designer applies rules which are based on experience rather than on general engineering standards.

According to its definition, reliability belongs clearly to the probabilistic framework in the frequentist context. On the one hand, this complicates probabilistic analysis of designs intended for limited production, since the fact that the product is only produced in limited quantity strongly complicates a decent a-posterior verification of the non-deterministic numerical predictions. Furthermore, for most designs intended for limited production, an unverifiably high reliability is requested (e.g. spacecraft). But an even bigger problem lies in probabilistic reliability analysis in the absence of trustworthy objective information. As discussed in section 2.2.1, while applying the probabilistic concept for the representation of subjective information is possible, results from such an analysis should definitely not be interpreted as indication for an absolute frequency of occurrence. The subjectiveness devaluates the use of the probabilistic results in a reliability context. This subjectiveness of (parts of) the information is not always detected. For instance, neglecting unknown correlation between properties by assuming them as independent is a common simplification that is sometimes implicitly made, but that can have important consequences. This implicit assumption of independence between probabilistic quantities was one of the important errors that were the source of the Challenger space shuttle disaster [41]. In this case, the impact of different extreme weather conditions on the launch was analysed for each condition individually beforehand. The impact of a combination of more than one of these events, however, was never checked. Although each of the events had a very low probability of occurring, the probability of their combination proved to be not simply a multiplication of the probability of the single events. The correlation between the conditions was clearly misjudged, leading to a plausible but unaccounted for weather situation with disastrous consequences.

The lack of credibility of numerical predictions of reliability is generally compensated by safety factors. However, one could argue that using these safety factors after applying sophisticated and computationally expensive numerical procedures is not a really economical situation. Much effort is spent on a numerical prediction, which, in the end, still has to be corrected based on practical experience. In this context, the non-probabilistic approaches could prove their value.

The application of the interval concept in numerical reliability studies is often referred to as *anti-optimisation*. This name stems from the fact that from all numerical models within the interval input boundaries, the one with the least favourable analysis result is the most interesting from reliability point of view. Finding this least favourable result is mathematically equivalent to performing a numerical optimisation aimed at the worst case result with respect to the input intervals.

The concept of anti-optimisation has been introduced as the basis for a non-probabilistic reliability framework [42]. This requires an evolution from a reliability concept as *probability of failure* towards *range of acceptable behaviour*. This means that the design must assure that the performance remains within an acceptable domain, without specifying a likelihood of failure. Reliability then becomes a crisp criterion distinguishing between either acceptable or unacceptable designs. The most important benefit of the anti-optimisation concept is that it broadens the objectivity of reliability studies to uncertain variabilities with known range, because the interval model perfectly represents these uncertainties without the need for subjective input. For instance, this enables a fast assessment of dimension tolerances on a design, without knowing the actual distribution of the dimension within the bounds of the prescribed tolerance. For some cases, it can be shown that the anti-optimisation procedure results in the same choice of design parameters as a probabilistic analysis if the required reliability tends to one [43]. The anti-optimisation in this case proves to be far less expensive in computation time.

The numerical implementation of the anti-optimisation approach is subject to an important requirement. Since the result of the analysis is the source of a crisp decision between acceptable and unacceptable designs, approximate results should always be kept on the safe side of the exact result. This means that if approximate solution procedures are used in the numerical implementation, they should guarantee conservatism in their result. On the other hand, this conservatism should not be excessively high in order for the result to be of any practical value.

Also the fuzzy concept can be usefully applied in a reliability framework to perform a possibilistic reliability analysis. In the interpretation of the membership function as a degree of possibility, the fuzzy outcome of an analysis could be used to define a possibility of failure. This possibility is clearly influenced by the subjectiveness that is implicitly incorporated in the fuzzy input of the analysis. This means that for the same problem, different analysts can and generally will end up with different possibilities of failure. This could be compensated by defining a personal threshold value for the allowed possibility of failure in the final decision on acceptable or unacceptable designs. However, due to the necessary amount of personal interpretation of the analyst, possibility of failure only has a relative value. Therefore, this approach is extremely difficult to standardise in a general reliability framework.

A different application of the fuzzy concept in reliability analysis is based on the use of the membership function as limit CDFs as explained in section 2.2.3. It was shown by FERRARI et al. [44] that if the input membership functions represent boundaries on the CDFs of the input parameters, the membership function resulting from fuzzy analysis on this input forms reliable boundaries on the actual CDF of the result. Therefore, the fuzzy result of a FFE analysis can be used to derive bounds on the probability of failure. A simple example illustrates this. Suppose that a FFE analysis results in a membership function $\mu_{\tilde{\lambda}}(\lambda)$ representing a crucial eigenfrequency of a design as illustrated in figure 8. Suppose furthermore that a crisp criterion states that the design is acceptable if this eigenfrequency is kept below the value λ^*. The fuzzy result envelopes the exact CDF of the eigenfrequency. This means that the bounds on the probability that the eigenfrequency of the design lies below λ^* can be derived from the fuzzy result. The probability interval is obtained from taking the value of the envelope curves at λ^* as indicated in the figure by $\underline{P'_f}$ and $\overline{P'_f}$. The most conservative statement resulting from the analysis is that the probability of failure equals $(1-\underline{P'_f})$ in the worst case.

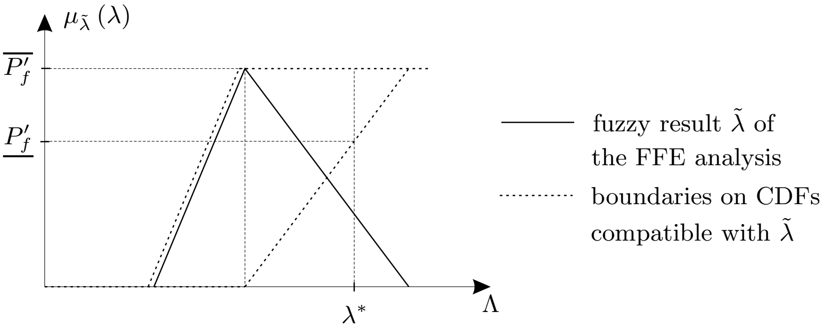

Figure 8: Example of the application of the fuzzy outcome of a FFE analysis to predict bounds on the probability of failure

It is clear that also the above non-probabilistic reliability methods are subject to the limitation that whenever there is subjective information involved in the problem definition, the results can not be interpreted as absolute measures of design quality. In an absolute reliability context, the amount of expert knowledge required in the distinction between a good or bad design is proportional to the amount of subjectiveness incorporated in the description of the non-determinism. Still, subjective analysis can be of great value when used in a relative framework, as for instance a design optimisation procedure. This will be discussed in the next section.

3.3.3 Numerical design optimisation

The principal goal of design optimisation is to define the best possible product under certain restrictions. These restrictions can be anything from manufacturing cost to limitations placed on physical properties of the design. The ingredients of the goal function and their relative weights determine the final result of the optimisation. Reliability can be used as an indication for the design quality, and therefore can be an important part of the goal function. When reliability is used in this manner, the demands on the objectivity are much lower than when it is used for absolute design assessment. A relative reliability improvement during an optimisation process can already be very valuable, even though the absolute reliability is only roughly approximated. This opens the door for any numerical tool that can handle subjective non-determinism, i.e. , the subjective probabilistic interpretation as well as the non-probabilistic approaches.

While applying subjective probability in this context is possible, it is not always the most advisable approach. In some cases, especially in design optimisation, a probabilistic reliability measure is not required. For instance, if the range on some parameters is all information that is available, placing subjective PDFs on these ranges only complicates the numerical problem, while it doesn't necessarily add any valuable meaning to the analysis. In that case, it doesn't really make sense to transform the problem to the probabilistic concept. Or as formulated by ROSS [45]: *Sometimes, striving for precision can be expensive, or adds little or no useful information, or both.* This indeed holds for the application of reliability calculations in an iterative optimisation procedure, where the numerical efficiency becomes very important. It is now discussed to what extend the non-probabilistic approaches can be considered as valuable alternatives for design analysis in an optimisation framework.

For the interval concept, the most useful application lies in modelling invariable uncertainties. Though they are assumed to be constant, they could play an important role during design optimisation. The analyst may ask the question whether the defined ranges for the invariable uncertainties result in an allowable range for the behaviour, without really being interested in the likelihood of occurrence within the defined interval bounds. Or, alternatively, the invariable uncertainty represents an open design decision, i.e. , a model property that has yet to be quantified, and the value of which will be optimised. Pure probabilistic analysis in both cases seems like an unnatural thing to do, since it requires information that is not available (probabilistic input) to produce information that is not requested (probabilistic output). The interval procedure is limited to the definition of the intervals on the uncertainties the analyst would like to take into account. Subsequently, the design can be assessed from an interval analysis by reassuring that the worst case output is still within the range of acceptable physical behaviour. This comes down to a worst-case oriented design optimisation.

A commonly formulated criticism on this approach is that the worst-case behaviour generally results from the combination of extremely rare events. Taking these combinations into account in a design assessment procedure could lead to severe over dimensioning. This criticism only holds if you can objectively verify the actual probability of occurrence of the model properties which are considered to be extreme events. But even more important, if you want to give a realistic weight to the actual occurrence of such an extreme combination of events, it is imperative to incorporate the exact mutual interdependence between these extreme events in the procedure, as discussed for the Challenger case in section 3.3.2. In such cases, worst case analysis could be a tool for identification of extreme events which lead to failure, without the need for a prediction of the actual probability of this extreme event. This identification should not necessarily lead to adapted designs and generally associated over dimensioning. In the Challenger case, accustomed launch protocols incorporating identification of possible disastrous extreme weather conditions, could already be of great value.

As discussed in the previous section, due to its implicit subjective nature, the value of fuzzy FE analysis as an absolute reliability analysis tool is rather limited. In an optimisation procedure, however, the complete process is generally conducted or followed up by one and the same analyst. This means that the subjective possibility measure can be interpreted in a consistent manner throughout the optimisation procedure. Therefore, the possibility of failure can be used as a quality measure in an optimisation procedure.

Apart from reliability optimisation, an important aspect of designing under uncertainty is to define a robust design, i.e. , a design whose critical properties have a minor sensitivity to changes in the uncertain influences like for instance external loading. Also in this context, the fuzzy approach can be of value. By placing fuzzy membership functions as loading factors on the crucial loading components, the sensitivity of some design quality indicators to these external influences can be analysed. Using this approach, the robustness of the design can be assessed by measuring the width of the resulting membership function on the critical design quality indicators.

Another practical approach of the fuzzy analysis is in the study and choice of tolerances placed on design dimensions. From the α-cut strategy, it is clear that the fuzzy FE analysis is actually a large-scale sensitivity analysis of the combined effect of the bounds defined on some interval design variables on critical design properties. By placing membership functions on the design properties subject to tolerances, the effect of their range on the design behaviour can be analysed. This can be helpful in defining tolerance intervals in the model. For instance, at a certain α-level, an allowable range could be identified in the fuzzy outcome of the analysis. The corresponding input intervals at this α-level can then be chosen as the set of tolerances on the analysed design properties. This procedure is clarified in figure 9, where the design specification is assumed to be an upper bound λ^* on an eigenfrequency. The analyst can control the analysis by defining the possibility distributions on the input according to personal preference or practical limitations. A different possibility distribution for the design variables will yield a different possibility distribution of the analysis result, and consequently also different tolerances for the design variables. The design based on these alternative allowable ranges, however, is equally safe. In this context, again, the possibility distribution is rather a useful tool to control the allowable range for the uncertainties than an absolute quality measure.

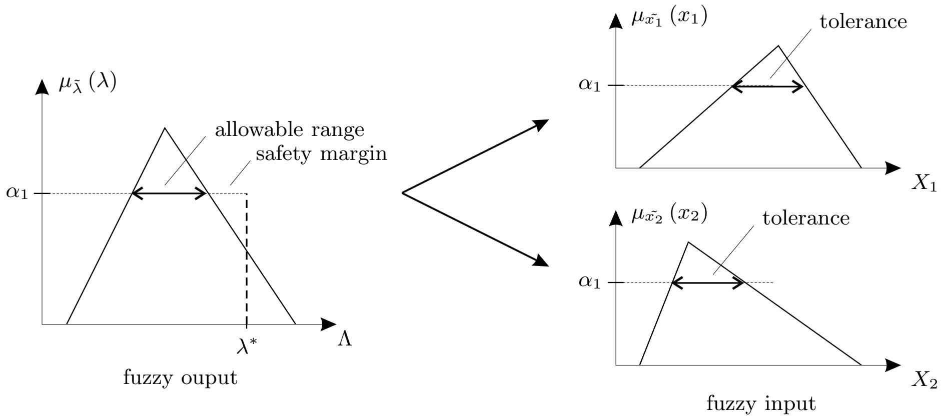

Figure 9: Illustration of the application of fuzzy concept for design tolerance analysis

4 Conclusion

The emerging non-probabilistic approaches are redefining the landscape for non-deterministic FE analysis. It is the aim of this paper to give insight into the possible useful applications of these approaches, referring to the generally accepted and widely adopted probabilistic approach.

It is first shown that a clear distinction can be made between different sorts of non-deterministic properties in a numerical model. The existing classification of uncertainties and variabilities is further subdivided in certain variabilities, uncertain variabilities and invariable uncertainties. Based on these different types of non-determinism, the applicability of the different non-deterministic concepts is analysed. It is concluded that the probabilistic approach remains the most interesting to tackle problems that are subject to complete and objective probabilistic influences. However, in the presence of uncertain quantities that require subjective information in order to be described numerically, the interval and fuzzy approach become increasingly interesting. Especially for uncertainties, the fuzzy concept is very appropriate because of its implicit subjective nature.

Concerning the numerical implementation of the non-probabilistic analysis procedures, an overview of available methodologies shows that there are different options for the IFE implementation. The interval arithmetic approach seems to be the least interesting, because of its high vulnerability to conservatism. The optimisation and the vertex approach are currently the most applied, although both suffer from specific restrictions and can become computationally expensive. The hybrid approach is promising, but needs further validation. Through the use of the α-level procedure, the FFE implementation can be directly derived from the corresponding IFE implementation.

Next, it is shown that in the framework of numerical design analysis, there generally is an evolution of the type of the non-determinism from uncertainty towards variability. Correspondingly, the non-probabilistic approaches tend to be most valuable in early design stages, whereas the probabilistic approach remains indispensable in later stages. This leads to the conclusion that the non-probabilistic approaches should be regarded as complementary rather than competitive to the probabilistic approach. However, not only the class of the non-deterministic properties encountered in the problem definition, but also the intended output determines to what extend the different non-deterministic approaches are appropriate numerical modelling tools for the treated problem. It is discussed how the non-probabilistic approaches can be of value in a typical design process. From the discussion, it has become clear that the value of the non-probabilistic approaches in an absolute reliability analysis is rather limited. It is concluded that non-probabilistic approaches will fail to convince in areas where absolute reliability measures are primordial. However, the application of subjective probability in this context has the same limitations. Absolute reliability analysis should always be performed in a frequentist interpretation, based on objectively available data. A small assumption in the probabilistic description of the input can lead to large misjudgement of the actual reliability of the design. This should always be kept in mind when applying numerical methods for absolute design reliability predictions.

Still, even limited or subjective information can be put to use in a design process, whenever the objective of the numerical analysis is a relative quality improvement. It is concluded that for numerical design optimisation, there are a number of useful applications for the non-probabilistic approaches. It is shown that they can be very valuable in robust design optimisation and tolerance analysis. Still, a profound study of the capabilities of the non-probabilistic approaches for design optimisation of industrially sized problems is required.

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