

Mid-frequency vibro-acoustic modelling: challenges and potential solutions

W. Desmet

K.U.Leuven, Department of Mechanical Engineering, division PMA
Celestijnenlaan 300 B, B-3001 Leuven (Heverlee), Belgium
e-mail: wim.desmet@mech.kuleuven.ac.be

Abstract

At present, the main numerical modelling techniques for acoustic and (coupled) vibro-acoustic analysis are based on element based techniques, such as the finite element and boundary element method. Due to the huge computational efforts, the use of these deterministic techniques is practically restricted to low-frequency applications. For high-frequency modelling, some alternative, probabilistic techniques such as SEA have been developed. However, there is still a wide mid-frequency range, for which no adequate and mature prediction techniques are available at the moment. In this frequency range, the computational efforts of conventional element based techniques become prohibitively large, while the basic assumptions of the probabilistic techniques are not yet valid.

In recent years, a vast amount of research has been initiated in a quest for an adequate solution for the current mid-frequency problem. This paper discusses the various methodologies that are being explored in this perspective. The main focus of this paper lies on the methodology that looks for deterministic techniques with an enhanced convergence rate and computational efficiency compared to the conventional element based methods in order to shift the practical frequency limitation towards the mid-frequency range. In this respect, special attention is paid to the wave based prediction technique for (coupled) vibro-acoustic analysis that is being developed at the KULeuven - Noise and Vibration Research group. The method is based on an indirect Trefftz approach. Various recent validations have revealed the beneficial convergence rate of this novel technique, thereby exhibiting its potential to comply with the mid-frequency modelling challenge.

1. Introduction

With the steadily increasing customer demands and competitive nature of the market, engineers face the challenging but complex problem of meeting the ever expanding but often conflicting design criteria. To accomplish this difficult task, the industry has become aware and convinced that a major imperative for getting insight in the sensitivity of the design criteria to the different design parameters and for making any advance beyond the very time-consuming and expensive prototype testing approach is to apply appropriate predictive engineering methods in all stages of the design process.

Along with the tightening of the legal regulations on noise emission levels and human exposure to noise, the acoustic properties of a product are often awarded a high priority on the list of design criteria. The acoustic properties are often determined by structure-borne noise, which is induced by the dynamic displacements of a mechanical structure. Since the pressure waves also cause a mechanical

load on the structure and hence influence the mechanical vibrations, there is a mutual fluid-structure coupling interaction. Especially for dynamic systems in which a small acoustic cavity is enclosed by a thin-walled mechanical structure, this structural-acoustic coupling interaction cannot be neglected. Vibro-acoustic effects play a dominant role in the sound quality of transportation means such as cars, aircraft, buses and trains, as well as in industrial, environmental and domestic noise levels. It becomes apparent from the above that there is a strong need for reliable prediction tools for vibro-acoustic analysis and optimal design of noise control measures.

At present, the main numerical prediction techniques for (vibro-)acoustic analysis are based on finite element, infinite element and boundary element methods ([1][2][3][4][5][6]). In these deterministic methods, the continuum domains or their boundary surfaces are discretized into small elements. Since the field variables within each element are described in terms of simple, but

approximating shape functions, a substantial amount of elements must be used in order to keep the approximation error within acceptable levels. In their use for dynamic problems, the element based methods exhibit numerical dispersion in that the wave number of the numerical solution differs from the wave number of the exact problem solution [7]. Since the dispersion error increases for increasing frequency, the mesh density and subsequent computational efforts, required for getting a reasonable level of accuracy, increase also with frequency. As a result, the use of element based methods for vibro-acoustic analysis is practically restricted to the low-frequency range. At higher frequencies, where structural and acoustic wavelengths are (substantially) smaller than the characteristic dimensions of the considered vibro-acoustic system, the numerical models become so large that, even with the nowadays available powerful computers, a prohibitively large amount of computing time would be needed.

For coupled vibro-acoustic problems, the conflict between computational effort and accuracy is even more pronounced than for uncoupled structural or uncoupled acoustic problems. First of all, the size of coupled prediction models is substantially larger, since a structural and an acoustic problem must be solved simultaneously to incorporate the acoustic pressure loading on the elastic structure and the continuity of the normal fluid and structural displacements along the fluid-structure coupling interface. Secondly, the numerical solution procedure for coupled vibro-acoustic models is less efficient than for uncoupled structural or uncoupled acoustic models, since coupled models, at least in their most commonly used acoustic pressure/structural displacement formulation [8], are no longer symmetric. Finally, the efficiency of the modal expansion method in reducing the model size is significantly reduced for coupled vibro-acoustic problems. The most appropriate mode selection for a modal expansion is the use of the modes of the coupled system. However, since coupled models are no longer symmetric, the calculation of coupled modes requires a non-symmetric eigenvalue calculation, which is very time consuming and which makes it a practically impossible procedure for many real-life vibro-acoustic problems. The most commonly used alternative is a modal expansion in terms of uncoupled structural and uncoupled acoustic modes, which result from symmetric and computationally efficient eigenvalue problems. However, the fact that uncoupled acoustic modes have a zero displacement component, normal to the fluid-structure coupling interface, implies that

a large number of high-order uncoupled acoustic modes is required to accurately represent the normal displacement continuity along the fluid-structure interface. Hence, the benefit of a computationally efficient construction of the modal base is significantly reduced by the smaller model size reduction, obtained with an uncoupled modal base. Besides the computational burden, an additional obstacle for these deterministic methods to be applied in the mid- and high-frequency range deals with the inevitable variability in the physical realisation of (vibro-acoustic) systems. Inherent, even small, variations in geometrical component dimensions, material properties and assembly tolerances imply large variations in the mid- and high-frequency response, so that one single deterministic prediction model with nominal system property values is no longer representative for the mid- and high-frequency dynamic behaviour of all possible physical realisations of the system under study. It is more appropriate to consider a (large) population of nominally identical systems and to provide information on the ensemble averaged dynamic response and the associated confidence levels.

In order to cope with the aforementioned limitations, major efforts have been devoted to the development of alternative, probabilistic prediction techniques. As such, Statistical Energy Analysis (SEA) [9] is the most wide-spread, industrially applied high-frequency modelling method. Complex vibro-acoustic systems are modelled as a composition of (weakly coupled) subsystems. The dynamic system response is described in terms of the frequency- and space-averaged energy response levels for each subsystem. These subsystem energy levels result from solving an SEA model that expresses energy balances for the various subsystems. In contrast with the element based methods, the size and subsequent computational effort of an SEA model are very small. Moreover, it is quite fair to assume that, due to the inherent complexity in practical systems, the frequency- and space-averaged results (for complex vibro-acoustic systems) give adequate information on the ensemble averaged results. However, one of the major assumptions of SEA is that each considered subsystem has a sufficient number of resonant modes ('high modal density') in the frequency range of interest. Therefore, SEA can only be applied for high-frequency (vibro-acoustic) modelling.

In-between the low-frequency application range of element based techniques and the high-frequency application range of probabilistic SEA techniques, there is still a wide mid-frequency range, for which no adequate and mature vibro-acoustic prediction techniques are available at the moment. Therefore, there is a strong need for novel prediction techniques that allow to narrow this currently existing mid-frequency twilight zone. In view of some characteristic properties of the mid-frequency dynamic behaviour of vibro-acoustic systems, those prediction techniques should meet the following requirements.

- The prediction techniques must be applicable for general, real-life vibro-acoustic engineering applications.
- In contrast with SEA models, in which all local information is lost, the prediction techniques should still provide detailed information on the spatial distribution of the mid-frequency response variables within the various components of the considered vibro-acoustic system.
- The prediction techniques must account for the effects of product and process variability and provide ensemble statistics of the mid-frequency dynamic behaviour.
- Many engineering structures, especially built-up structures such as vehicles, trains, truck cabins, etc., have a particular mid-frequency dynamic behaviour: parts of the structure consist of stiff and strongly connected components that still exhibit long wavelength dynamic deformations in the mid-frequency range, while other parts exhibit already a highly resonant behaviour with short wavelength deformations. The novel prediction techniques should enable a hybrid modelling approach in which the long wavelength components are modelled deterministically, while the short wavelength components are described in a probabilistic manner.

2. Possible methodologies

In recent years, a vast amount of research activities have been initiated in a quest for an adequate solution for the current mid-frequency problem. Three major methodologies can be adopted in this framework.

- A first methodology consists of developing a deterministic technique with an enhanced computational efficiency compared to the conventional element based techniques. In this

way, the practical frequency limitation can be shifted towards the mid-frequency range and statistical information on the prediction results can be obtained through efficient Monte Carlo simulations or through special-purpose stochastic procedures.

- A second methodology consists of improving the applicability of probabilistic techniques towards mid-frequency modelling by relaxing some of the most stringent assumptions of conventional SEA, as is proposed e.g. in the Wave Intensity Analysis [10] and the Statistical Modal Energy Distribution Analysis [11]. In an attempt to provide information on the spatial energy distribution within the various subsystems, research efforts are being devoted to the Energy Flow Analysis (EFA) and its numerical implementation through the Energy Finite Element Method (EFEM) ([12][13][14][15][16][17]). The method is based on the extension of the basic SEA equations for finite modal subsystems into differential terms and enables the description of the dissipation and conduction of energy within each subsystem. Since this description is formally equivalent to the steady-state equation of heat flow in solids, these methods are called 'thermal' methods. A major advantage of this similarity with thermal problems is that the energy distribution and energy flow within the basic components can be easily computed with readily available finite element codes for thermal computations. Because of the finite element formulation, the database for these methods is similar to the database needed for a conventional (low frequency) FEM calculation. In this way, a low frequency analysis by means of FEM can be easily extended to an analysis in higher frequency bands. Other investigations include the development of envelope methods that employ suitable variable transformations to obtain new dynamic equations that govern the moderately varying envelope of the high-frequency dynamic deformations [18].
- A third methodology is a hybrid approach, in which deterministic and probabilistic techniques are combined. A major obstacle to SEA is that several engineering structures consist not only of parts that have many resonances and which are distinctly different 'elements' but also of stiff and strongly connected parts, most often designed to support and secure the structure. Often, the greatest responses are found in the flexible parts with the transmission of vibration in the supporting structure. An FE analysis of

such structures is not possible due to the computational burden. On the other hand, SEA does not apply for strongly coupled and stiff structures with few resonances. Consequently, the mid- and high-frequency prediction of noise and vibration in built-up structures such as vehicles requires a hybrid approach. One such hybrid approach is the ‘fuzzy structure theory’ ([19][20][21]). In this theory, the master structure is modelled deterministically. The influence of fuzzy attachments on the master structure is, upon a stochastic assumption, modelled as an increase of inertia and damping. Langley and Bremner [22] developed a hybrid approach based on a wavenumber partitioning scheme. The subsystems in a system are partitioned into those that exhibit long wavelength ‘global’ behaviour (and possess no dynamic uncertainty) and those that exhibit short wavelength ‘local’ behaviour (and possess significant dynamic uncertainty). The long wavelength global behaviour is then modelled deterministically using FE, while the short wavelength local behaviour is modelled statistically using SEA. There is clearly an interaction between the two partitions of the model: the effect of the ‘local’ degrees of freedom on the ‘global’ degrees of freedom is accounted for via an approach which is analogous to fuzzy structure theory, while the SEA equations for the ‘local’ degrees of freedom take into account the power input from the ‘global’ degrees of freedom. Some recent applications of this hybrid approach include the dynamic analysis of a panel-frame structure ([23]) and of a cavity backed plate ([24]). Zhao and Vlahopoulos [25][26] propose another hybrid approach by using a deterministic FE model for the global long wavelength components and an energy finite element model for the local short wavelength components. The global FEM and local EFEM equations, together with the coupling interface equations, are solved simultaneously through an iterative but computationally efficient process. So far, this approach has only been validated for co-linear beam networks.

The remainder of this paper will focus on some recent developments within the first methodology, i.e. the development of deterministic techniques with an enhanced convergence rate and computational efficiency compared to the conventional element based methods.

3. Deterministic approaches

3.1 Introduction

In a deterministic model, the solution for the governing partial differential equation (PDE) is sought at each position in the problem domain. In the case of time-harmonic acoustic problems, for instance, the steady-state pressure p in domain Ω is governed by the Helmholtz equation

$$\mathcal{L}p = f \quad (1)$$

where f is a known distribution of acoustic excitations and where $\mathcal{L} = -\nabla^2 - k^2$, with $k = (\omega/c)$ the acoustic wave number, being the ratio between the circular frequency of excitation ω and the speed of sound c . To uniquely define the pressure field at any position in the problem domain Ω , some boundary conditions (BC) must be specified at the boundary surface $\Gamma = \Gamma_d \cup \Gamma_h$ of the problem domain,

$$p = d, \quad \text{on } \Gamma_d \quad (2)$$

$$p_{,n} + j\alpha kp = h, \quad \text{on } \Gamma_h \quad (3)$$

where $.,_n$ denotes the gradient, normal to the boundary surface, and where d , h and α are known functions on the respective parts of the boundary surface.

This PDE can be interpreted in a weak form, i.e. the solution is sought in a trial space $P = \{p \mid p \in H^1(\Omega), p = d \text{ on } \Gamma_d\}$ such that the weak formulation

$$a(w, p) + j\alpha k(w, p)_{\Gamma_h} = (w, f)_{\Omega} + (w, h)_{\Gamma_h} \quad (4)$$

is satisfied for all functions of a test space $W = \{w \mid w \in H^1(\Omega), w = 0 \text{ on } \Gamma_d\}$. In this weak form, $(.,.)_{\Omega}$ denotes the $L_2(\Omega)$ inner product and $a(w, p) := (\nabla w, \nabla p)_{\Omega} - k^2(w, p)_{\Omega}$.

In the classical finite element method (FEM), the trial space P and the test space W are replaced by finite dimensional approximation spaces P^h and W^h and the finite element solution p^h is then given as the element of the finite dimensional trial space P^h that satisfies the weak formulation for all test functions of the finite dimensional test space W^h . The approximation spaces P^h and W^h result from a discretization of the problem domain Ω into a set of nonoverlapping element domains Ω^e . Within each

element domain, some shape functions N_a^e are defined and the FE approximation is then defined as a linear combination of these shape functions,

$$p^h = \sum_a N_a^e \cdot p_a \quad \text{in } \Omega^e \quad (5)$$

The coefficients p_a represent the nodal degrees of freedom at some discrete, nodal positions of the element. The shape functions are based on low-order polynomial interpolation functions and are defined such that the elements are conforming, which means for the case of the Helmholtz equation that the FE pressure approximation and its first-order derivatives must exist within each element and that the pressure approximation must be continuous across the element boundaries, but that its (first-order) derivatives may be discontinuous across the element boundaries. By adopting a Galerkin scheme, i.e using the same shape functions for both the trial and test spaces P^h and W^h , the weak formulation (4) yields the acoustic Galerkin FE model

$$A \cdot p_a = (K_a + j\omega C_a - \omega^2 M_a) \cdot p_a = f_a \quad (6)$$

where the system matrix A consists of the sparsely populated, banded, symmetric stiffness matrix K_a , mass matrix M_a and damping matrix C_a and where vector p_a comprises the unknown nodal degrees of freedom of the FE model.

Since the field variables within each element are described in terms of simple, but approximating shape functions, a substantial amount of elements must be used in order to keep the prediction error within acceptable levels. In its application for dynamic wave problems, an upper bound for the relative prediction error of the FE method can be defined ([27][28]),

$$\varepsilon \leq C_1 \left(\frac{kh}{p} \right)^p + C_2 kL \left(\frac{kh}{p} \right)^{2p} \quad (7)$$

where C_1 and C_2 are constants, where h is the finite element size, p is the order of the polynomial shape functions and L is a characteristic length of the problem domain. The first term in this prediction error is related to the approximation error, which is the difference between the exact wave and its interpolant. The second term is related to the pollution error, which is the difference between the interpolant and the finite element wave and which is mainly determined by numerical dispersion. Indeed,

it is well known ([7][29][30]) that element based methods exhibit numerical dispersion in that the wave number of the numerical solution differs from the wave number of the exact problem solution.

For low-frequency application, the approximation error dominates the prediction error. As can be seen from (7), the prediction error can be limited by keeping kh constant. This has led to the rule of thumb stating that a fixed number of elements per wavelength (say e.g. 6 to 10 linear elements per wavelength) are required to keep the prediction error within acceptable limits. Since wavelengths decrease for increasing frequency, the mesh density and subsequent computational efforts, required for getting a reasonable level of accuracy, increase also with frequency. Another way to limit the prediction error is to increase the spectral order p through the use of higher-order finite elements. As discussed in [31],[32], increasing the order of the shape functions (for a given element size) results in an increase of the accuracy, at the expense, however, of an increase in the matrix bandwidth, which on its turn induces an increase of the computational effort. As a result, a general trend is observed that linear elements ($p=1$) are more efficient when lower accuracy is sufficient, while quadratic elements ($p=2$) are more efficient when higher accuracy is desired.

At higher frequencies, the pollution error becomes dominant. It follows from (7) that, even by keeping kh constant, the prediction error still grows with wavenumber, and hence with frequency. As a result, for higher-frequency applications, the element size h must be kept very small, and, hence, the resulting FE models become so large that a prohibitively large amount of computing time and memory resources would be needed. In this way, the FEM for dynamic problems is practically restricted to low-frequency applications, leaving, as stated recently by Zienkiewicz [33], short wavelength dynamic problems amongst the currently unsolved problems of the finite element method.

3.2 efficient solvers

One possible way to extend as much as possible the frequency range of application is to develop efficient solvers for large FE models (6), that can possibly be implemented in a parallel computational environment.

Direct solution methods, which are based on Gaussian elimination, are reliable and provide an a priori estimate of the computational effort required. However, their computational cost grows rapidly with the number of equations, even when special-

purpose solvers are used that exploit the symmetry and banded character of FE matrices.

As an alternative to direct solution methods, gradient-type iterative solvers, which are based on working with sequences of orthogonal vectors, are among the most effective iterative procedures for solving large sparse linear systems and can also be very attractive for large dense matrix problems. For non-Hermitian indefinite systems, as is the case for acoustic FE and BE models, two major approaches can be identified ([34][35]). The Generalized Minimal Residual procedure (GMRES) computes and stores explicitly a sequence of orthogonal vectors that can be combined using a least-squares solve to generate iterates that have a minimal residual in the subspace spanned by the vectors. A major disadvantage of this procedure is that the entire sequence of vectors must be stored. In the Quasi-minimal Residual method (QMR) using the unsymmetric Lanczos process to generate two sequence of vectors that satisfy a bi-orthogonality condition. Iterates are obtained by enforcing a Galerkin-type of condition to the residuals. This method involves only a limited amount of work and storage, at the expense of possible irregular convergence.

In contrast to direct methods, the performance of iterative solvers is highly problem-dependent and the convergence of some schemes may deteriorate severely under certain conditions, hindering the advance determination of execution-time requirements. Their performance is particularly sensitive to the condition number of the coefficient matrix A in (6) and the distribution of its eigenvalues. This sensitivity is often addressed by preconditioning and scaling, which can lead to substantial improvement in convergence rates. A preconditioner M is employed such that $M^{-1}A$ is better conditioned and that linear systems with coefficient matrix M are easy to solve. Hierarchical elements play an important role in the framework of the construction of preconditioners ([36]). Hierarchical basis functions are associated with a multilevel splitting of the FE-mesh: basis functions on the coarsest level, level 1, consist of the nodal basis functions and the hierarchical basis at any level j consists of the nodal basis functions corresponding to nodes in that level which are not present in any of the coarser levels, together with the hierarchical basis for level $j-1$. Hierarchical elements are better conditioned, but the matrices are less sparse, since the support of the hierarchical basis functions is less compact. However, the explicit computation of the hierarchical coefficient matrices can be avoided by using a transformation

which maps the hierarchical basis representation of a function to its nodal representation. This transformation can be used to construct a preconditioner for iterative solution procedures. The QMR with a hierarchical basis preconditioner seems to perform very efficiently (for uncoupled 2D problems).

In the case of multiple load cases, for instance, an FE model of type (6) must be solved with multiple right-hand sides. Direct methods are well suited when the coefficient matrix can be stored and factorized efficiently. Since the factorization is the dominating cost in direct methods, multiple right-hand sides can be handled easily by performing the factorization only once and then obtaining multiple solutions at a cost of two backsolves per right-hand side. The aforementioned iterative procedures are tailored to the solution of linear systems with a single right-hand side. A block variant of the Krylov subspace method has been proposed as an efficient iterative procedure when there are multiple right-hand sides ([37]).

Often the dynamic solution has to be calculated for multiple frequencies, and it is sometimes sufficient to evaluate the solution over restricted regions of the spatial domain (such as e.g. over an enclosing surface in the near-field of a sound source) instead of over the entire problem domain. For these cases, some efficient solution strategies have been formulated ([38][39]), which adopt a Padé-approximation strategy, that constructs rational functions that coincide in a specified number of terms with the power series expansion of the function to be approximated.

3.3 adaptivity

As described before, the main reason for the large size of element based models is the fine element discretization that is needed for an accurate representation of the dynamic response. The construction of such an element discretization (mesh generation) is based on the engineering judgement and experience of the analyst and performed by an automatic mesh generator. This usually results in a mesh of fairly uniform density, i.e. a mesh in which most elements have a similar geometrical size and in which the polynomial shape functions of the parent elements are of the same (low) order. However, since the spatial variation of the dynamic response often varies within the considered domain, a fine discretization is only required in those areas of the domain where the response has a high spatial variation, while a coarse discretization is allowed in the areas where the response has a low spatial

variation. Hence, a uniform mesh density is not always the most efficient discretization.

To optimise the discretization, i.e. to obtain a solution of given accuracy with a minimum number of degrees of freedom, some adaptive mesh refinement techniques have been developed in order to get an element size distribution such that the prediction error is equidistributed among the elements. In these techniques, the considered domain is initially discretized into a coarse mesh of fairly uniform density. A local a posteriori error estimator evaluates the distribution of the resulting prediction errors over the domain. In areas of large error, the adaptive strategy refines the discretization by locally dividing the elements into several smaller elements (h-method) or by locally increasing the order of the polynomial shape functions in the elements (p-method) or a combination of both (hp-method). After some iterations of this procedure, a mesh is obtained, in which a fine discretization is only used in those areas where it is really necessary. Some dedicated local error estimators (see e.g. [40],[41],[42]) have been developed for acoustic problems. Several references can be found on the use of h-adaptive (see e.g. [10],[34],[40],[43],[44],) and hp-adaptive methods ([28],[45]) for Helmholtz problems. In the framework of a p-adaptive strategy, the aforementioned hierarchical elements are often used (see e.g. [2][34]), since the mass and stiffness matrices of order p can be reused when increasing the order to $p+1$. To illustrate the enhanced computational efficiency of adaptive techniques, it is reported in [40] that speed-ups of a factor of 20 can be achieved by using an h-adaptive strategy instead of using a uniform mesh distribution.

Since these techniques ensure that there is no waste of computational effort, they could also be very useful for the computationally highly expensive coupled vibro-acoustic problems. They are, however, not yet applied for this type of problems, at least to the author's knowledge.

3.4 domain decomposition

Another possible approach to reduce the computational efforts, involved with solving large dynamic FE models, is domain decomposition, in which the size of the original FE model is substantially reduced or where solving one large problem is replaced by solving many smaller subproblems. These methods are perfectly suited for implementation in a parallel computational environment.

In the *component mode synthesis* (CMS) approach, the structure is divided into components and each component is then analysed independently for natural frequencies and mode shapes. The component mode shapes are then assembled to give displacement shapes of the original structure. These mode shapes are not eigenvectors of the original structure, but they are only used to transform the original degrees of freedom into generalised degrees of freedom. This may reduce significantly the size of the original system matrix. A survey of CMS methods can be studied in [46]. The difference in these methods lies in how the modes are combined over the boundary between the subdomains. A commonly used method is the Craig-Bampton approach ([47][48]), where two sets of substructure modes are used to transform the subcomponent degrees of freedom: component normal modes, i.e. normal modes with all degrees of freedom of the interface held fixed, and constraint modes, i.e. deflections induced in the component structures by applying a unit displacement to one interface degree of freedom while all other interface degrees of freedom are held fixed. A significant model size reduction can be obtained with the Craig-Bampton CMS method. It should be noted, however, that, since there is one constraint mode needed for each interface degree of freedom, the cost of this method could still be prohibitive for a sufficiently fine mesh. In this respect, the Automated Multi-level Substructuring method (AMLS) has been proposed recently [49][50]. The first step is to partition an FE model automatically into substructures on a number of levels, based on the sparsity structure of the system matrices. Substructures on the lowest level consist of a small number of finite elements. These 'child' substructures are assembled together to form 'parents', which are assembled together to form 'grandparents' and so on until the model of the complete structure has been assembled. This results in a tree topology for the substructures. For large FE models, substructures typically number at least in the thousands. Within a substructure on the lowest level, degrees of freedom are partitioned into two sets. One set consists of both 'interface' and 'forced' degrees of freedom, i.e. those that are shared with adjacent substructures at interfaces, and those that correspond to nonzeros in the force vector. These are called 'shared' degrees of freedom, since the substructure shares these directly with its environment, in a general sense. The set of remaining degrees of freedom consists of 'local' degrees of freedom; they are not excited directly, but only through coupling with the shared degrees of freedom. The lowest-level substructure FE

models are reduced using a Craig-Bampton CMS transformation. Once lowest-level 'child' substructures have been transformed, they are assembled together to form 'parent' substructures on the next level. Since local child degrees of freedom are represented implicitly in terms of shared degrees of freedom by default, parent degrees of freedom are considered to consist only of child shared degrees of freedom. From these, local and shared degrees of freedom for parents are identified and parent models are transformed as the child substructure models were. Parent 'modes' are not intended to be accurate approximations of physical modes of vibration; instead, they constitute an efficient reduced subspace with useful orthogonality properties. The assembly procedure to form higher-level substructures and the Craig-Bampton transformation continue until a model for the entire structure has been assembled. If all 'forced' degrees of freedom are included as 'shared' degrees of freedom at all levels, all degrees of freedom that have been 'local' for lower-level substructures implicitly depend quasistatically on forced and other shared degrees of freedom in the last assembled model. As a result, static response of the structure is represented exactly without using any modes of lower-level substructures. Once a model for the entire structure has been assembled, its degrees of freedom that are not forced are treated in the same manner as local degrees of freedom for lower-level substructures, and forced degrees of freedom are treated as the only shared degrees of freedom. A partial eigensolution is found and the model is transformed as described above. After this transformation, the only remaining explicit degrees of freedom are forced degrees of freedom. One last eigensolution is found, in which only forced degrees of freedom appear. The response of the structure can then be expressed entirely in terms of modes of substructures on all levels. AMLS requires much lower disk space requirements than the conventional modal synthesis approach, since in the conventional modal approach, thousands of eigenvectors and iteration vectors, expressed in all FE degrees of freedom, must be stored to disk, while in AMLS it is not necessary to produce eigenvectors in all FE degrees of freedom. The partial eigensolutions are economical because the substructure eigenvalue problems are very small, and their eigenvalues are typically well separated, resulting in rapid convergence. In this way, significant savings in computational effort can be achieved with AMLS, as is illustrated in [51] for real-life automotive engineering applications.

An alternative to component mode synthesis is *Guyan reduction* ([52]). In this approach, certain degrees of freedom, the masters, are to be retained and the slave degrees of freedom are to be removed by condensation. In static condensation, it is assumed that inertia forces on slave degrees of freedom are less important than elastic forces transmitted by the master degrees of freedom. In this way, the performance of the method is strongly dependent on the appropriate choice of master degrees of freedom.

The *Finite Element Tearing and Interconnecting method* (FETI) was introduced in [53]. It is a two-level domain decomposition method for solving iteratively large-scale systems of equations arising from a finite element discretization. The basic idea of the method is to decompose the original domain into non-overlapping subdomains and to use Lagrange multipliers to enforce the continuity conditions along the interfaces between the various subdomains. In each iteration step, one (small) independent local problem per subdomain is solved directly. A global problem, evaluating the residuals on the interface continuity constraints, is then solved iteratively (with preconditioning) and drives the next iteration step until the constraint residuals tend to zero. Its application for acoustic problems is reported e.g. in [54][55][56][57][58]. Its application for plate and shell dynamic modelling is reported in [59][60]. Recently, the method has also been applied for coupled fluid-solid problems ([61]).

A similar iterative domain decomposition approach is presented in [62][63], however without the use of Lagrange multipliers, but where the continuity conditions for the various subproblems in a next iteration are obtained from the subproblem results of the previous iteration.

3.5. knowledge-based FE methods

The key property of knowledge-based FE methods is that they try to incorporate a priori knowledge about the considered (dynamic) problem into the numerical model in order to reduce the involved numerical dispersion, so that the finite elements are capable of containing many wavelengths. Various possible modelling approaches are currently being explored in the research community. Some of the most prominent ones are reported in this section.

3.5.1 stabilized techniques

Most fluids in an acoustic problem are non-dispersive in that the speed of sound c is

independent of the frequency ω , resulting in an acoustic wave-number k ($=\omega/c$) that is proportional and an acoustic wavelength λ ($=2\pi/k$) that is inversely proportional to the frequency. An entirely dispersion-free representation of an acoustic wave cannot be obtained from a conventional Galerkin finite element formulation and this dispersion error gets worse for increasing frequency.

To reduce the dispersion error, Harari and Hughes [64] proposed the Galerkin least-squares (GLS) finite element method for solving one-dimensional Helmholtz problems, which has been extended later on for solving two-dimensional [65] and three-dimensional [66] uncoupled acoustic problems. The method is based on a formulation, in which a least-squares form of the residual of the governing Helmholtz equation is added to the conventional Galerkin weighted residual formulation (4). The solution is sought in the trial space P such that the weak formulation

$$\begin{aligned} a(w, p) + j\omega k(w, p)_{\Gamma_h} + (\sqrt{\tau_f} \nabla \mathcal{L}w, \sqrt{\tau_f} \nabla \mathcal{L}p)_{\Omega} \\ = (w, f)_{\Omega} + (w, h)_{\Gamma_h} + (\sqrt{\tau_f} \nabla \mathcal{L}w, \sqrt{\tau_f} \nabla f)_{\Omega} \end{aligned} \quad (8)$$

is satisfied for all functions of a test space W . In this weak form, Ω' denotes the union of the finite element interiors. The least-squares form contains the mesh parameter τ_f , which can be determined such that the dispersion error is minimized, i.e. by matching the numerical wavenumber to the continuous wavenumber through the dispersion relation obtained from a discrete Fourier transform of the difference stencil of the GLS discretization. This mesh parameter can be thought of as a local quantity to be defined consistently on the element level in terms of element sizes. It may vary from element to element but is constant over the element interiors. Since the additional terms in the weak formulation are evaluated over element interiors, the lack of global smoothness of the FE functions across the element boundaries creates no problems, so that the consistency, inherent in the parent Galerkin method, is preserved.

On regular grids in one dimension, the GLS can eliminate the phase error and provide a dispersion-free solution. For two- and three-dimensional acoustic problems, however, elimination of pollution error is not possible using any generalized FEM. The stabilizing effect of the GLS-FEM is sensitive to the direction of the exact solution; the error can be reduced in certain "preferred" directions, but the stabilized approach, in general, has little effect if the exact solution doesn't have

dominant components in one of these directions. It should be noted also that, at least in its current stage of development, the use of this stabilized FE method is restricted to regular grids.

In the Galerkin gradient least-squares (GVLS) finite element method [34], the solution is sought in the trial space P such that the weak formulation

$$\begin{aligned} a(w, p) + j\omega k(w, p)_{\Gamma_h} + (\sqrt{\tau_f} \nabla \mathcal{L}w, \sqrt{\tau_f} \nabla \mathcal{L}p)_{\Omega} \\ = (w, f)_{\Omega} + (w, h)_{\Gamma_h} + (\sqrt{\tau_f} \nabla \mathcal{L}w, \sqrt{\tau_f} \nabla f)_{\Omega} \end{aligned} \quad (9)$$

is satisfied for all functions of a test space W . Again, the mesh parameter τ_f is determined such that the dispersion error is minimized. It is indicated in [67] that, in contrast with the GLS, this GVLS deteriorates for acoustic wave propagation problems on non-uniform meshes. However, as indicated in [68], the gradient least-squares approach seems to be beneficial for the modelling of the propagation of mechanical waves in an elastic medium.

For (two-dimensional) time-harmonic coupled structural-acoustic problems, it is shown in [69] how the GLS method can be used for the acoustic part, while conventional Galerkin FEM is used for the structural part. In [70],[71], both parts are modeled using a stabilized FE method, in that the GLS method is used for the acoustic part, the GVLS method for the longitudinal structural vibrations and the GLS method for the transverse plate vibrations.

In the GLS and GVLS methods, the modified operator stems from a modified variational form; this form is subsequently discretized by the standard FEM approach, i.e. modifications are introduced on the functional level. In the quasi-stabilized FEM ([72][73]), however, the modification of the discrete operator are imposed on the algebraic level directly into the dynamic stiffness matrix. The QSFEM (in 1D) uses linear shape functions but has nodal values that are obtained from analytic shape functions, so that the solution is exact at the nodal points. In this way, the QSFEM can eliminate the phase error in 1D on any non-uniform mesh. For two-dimensional problems, the pollution error is nowhere completely eliminated, but is minimized for all possible directional components of the exact solution. As indicated in [7], the QSFEM is most effective, compared to other stabilized methods but complicated in the general setting, due to the complicated computations that are needed in order to derive the method for complicated grids and the

difficulty to adapt it to non-uniform meshes and irregular boundary conditions.

3.5.2. multi-scale techniques

In multi-scale techniques, as proposed in [74],[75], the problem solution is assumed to consist of both coarse and fine scales, say $p = \bar{p} + p'$, in which \bar{p} and p' represent the coarse scale and the fine scale, respectively. The coarse scale solution is resolvable numerically (using standard finite elements on an unstructured grid of non-overlapping element subdomains), while the fine scale solution p' is determined analytically, eliminating it from the problem for \bar{p} . The characteristic length scale of the finite element grid is much larger than the fine scales, hence it is referred to them as subgrid scales. On the other hand, the characteristic length scale of the grid is assumed small compared with the coarse scales. The two scales \bar{p} and p' may overlap or be disjoint, and p' may be globally or locally defined, but the effect of p' on the problem for \bar{p} will always be non-local.

In its application for Helmholtz problems (see e.g. [76]), the weak formulation (4) must be reformulated in terms of the resolvable coarse scales only. To achieve this multi-scale formulation, consider an element discretization of the problem domain Ω . Ω' is the union of element interiors and Γ' is the union of element interfaces (without the problem boundary Γ). Let $p = \bar{p} + p'$ and $w = \bar{w} + w'$, so that $P = \bar{P} \oplus P'$ and $W = \bar{W} \oplus W'$, where \bar{P} (resp. P') is the trial solution space for the coarse (resp. fine) scales and \bar{W} (resp. W') is the test function space for the coarse (resp. fine) scales. By assuming

$$\bar{p} = d \text{ on } \Gamma_d \quad \forall \bar{p} \in \bar{P}, \quad (10)$$

$$p' = 0 \text{ on } \Gamma \quad \forall p' \in P', \quad (11)$$

$$\bar{w} = 0 \text{ on } \Gamma \quad \forall \bar{w} \in \bar{W}, \quad (12)$$

$$w' = 0 \text{ on } \Gamma \quad \forall w' \in W' \quad (13)$$

the original weak formulation of the Helmholtz problem

$$a(\bar{w} + w', \bar{p} + p') + j\alpha k(\bar{w} + w', \bar{p} + p')_{\Gamma_h} = (\bar{w} + w', f)_{\Omega} + (\bar{w} + w', h)_{\Gamma_h} \quad \forall w \in W \quad (14)$$

can be transformed into two sub-problems

$$a(\bar{w}, \bar{p}) + j\alpha k(\bar{w}, \bar{p})_{\Gamma_h} + a(\bar{w}, p') + j\alpha k(\bar{w}, p')_{\Gamma_h} = (\bar{w}, f)_{\Omega} + (\bar{w}, h)_{\Gamma_h} \quad (15)$$

$$a(w', \bar{p}) + j\alpha k(w', \bar{p})_{\Gamma_h} + a(w', p') + j\alpha k(w', p')_{\Gamma_h} = (w', f)_{\Omega} + (w', h)_{\Gamma_h} \quad (16)$$

By integration by parts and by defining $[[\cdot]]$ as the jump in a variable across element boundaries, the first sub-problem can be written as

$$a(\bar{w}, \bar{p}) + j\alpha k(\bar{w}, \bar{p})_{\Gamma_h} + (\mathcal{L}\bar{w}, p')_{\Omega} + ([[\bar{w}]], p')_{\Gamma'} + j\alpha k(\bar{w}, p')_{\Gamma_h} = (\bar{w}, f)_{\Omega} + (\bar{w}, h)_{\Gamma_h} \quad (17)$$

and the second sub-problem can be written as

$$(w', \mathcal{L}\bar{p} + \mathcal{L}p')_{\Omega} + (w', [[\bar{p}]] + [[p']])_{\Gamma'} + (w', \bar{p}_{,n} + j\alpha k\bar{p} + p'_{,n} + j\alpha kp')_{\Gamma_h} = (w', f)_{\Omega} + (w', h)_{\Gamma_h} \quad (18)$$

By defining the orthogonal projection $\Pi': W \rightarrow W'$ of W onto W' , the second sub-problem can be defined as

$$\begin{aligned} \Pi' \mathcal{L}p' &= -\Pi'(\mathcal{L}\bar{p} - f) \text{ in } \Omega \\ p' &= 0 \text{ on } \Gamma_d \\ p'_{,n} &= -j\alpha kp' - (\bar{p}_{,n} + j\alpha k\bar{p} + h) \text{ on } \Gamma_h \end{aligned} \quad (19)$$

The relevant fine-scale Green's function problem corresponding to this second subproblem is

$$\begin{aligned} \Pi' \mathcal{L}g'(\mathbf{x}, \mathbf{x}_0) &= \Pi' \delta(\mathbf{x} - \mathbf{x}_0) \text{ in } \Omega \\ g'(\mathbf{x}, \mathbf{x}_0) &= 0 \text{ on } \Gamma_d \end{aligned} \quad (20)$$

$$g'(\mathbf{x}, \mathbf{x}_0)_{,n_x} = -j\alpha kg'(\mathbf{x}, \mathbf{x}_0) \text{ on } \Gamma_h \quad (21)$$

so that the general solution for the fine scale is then

$$\begin{aligned} p'(\mathbf{x}_0) &= - \int_{\Omega'} g'(\mathbf{x}, \mathbf{x}_0) \cdot (\mathcal{L}\bar{p} - f)(\mathbf{x}) \cdot d\Omega_x \\ &\quad - \int_{\Gamma'} g'(\mathbf{x}, \mathbf{x}_0) \cdot ([[\bar{p}]]) (\mathbf{x}) \cdot d\Gamma_x \\ &\quad - \int_{\Gamma_h} g'(\mathbf{x}, \mathbf{x}_0) \cdot (\bar{p}_{,n} + j\alpha k\bar{p} - h)(\mathbf{x}) \cdot d\Gamma_x \\ &= M(\bar{p}, f) \end{aligned} \quad (22)$$

Since the fine scale solution represents the error in the coarse scale solution, the above representation of

the fine-scale solution in terms of the distributional coarse-scale residual and the fine-scale Green's function is a paradigm for a posteriori error estimation.

Using the above defined integral operator M , the first sub-problem may then be written completely in terms of the coarse scales

$$\begin{aligned}
 & a(\bar{w}, \bar{p}) + j\alpha k(\bar{w}, \bar{p})_{\Gamma_h} \\
 & + (\mathcal{L}\bar{w}, M(\bar{p}, f))_{\Omega'} + (\llbracket \bar{w}, n \rrbracket, M(\bar{p}, f))_{\Gamma'} \\
 & + j\alpha k(\bar{w}, M(\bar{p}, f))_{\Gamma_h} \\
 & = (\bar{w}, f) + (\bar{w}, h)_{\Gamma_h}
 \end{aligned} \tag{23}$$

This is the variational multi-scale formulation of the Helmholtz equation; an equation for \bar{p} that exactly accounts for the effect of p' on \bar{p} .

When implementing a finite dimensional approximation of the variational multiscale formulation, two important problems must be encountered. On the one hand, an a priori knowledge of the fine scale Green's function g' is required, and, on the other hand, the integral operator M acting on \bar{p} will couple all the degrees of freedom for p' in the problem domain Ω .

In the numerical implementation, the spaces \bar{P}^h and \bar{W}^h , which are the finite-dimensional counterparts of \bar{P} and \bar{W} , may be chosen to precisely allow for piecewise linear, bilinear and trilinear FE interpolation functions in one, two and three dimensions respectively. It is further assumed that the subgrid scales vanish on the element boundaries, i.e. $p'=0$ on $\Gamma \cup \Gamma$ and $w'=0$ on $\Gamma \cup \Gamma$. Under these assumptions, the subgrid finite element model for the Helmholtz equation becomes

$$\begin{aligned}
 & a(\bar{w}^h, \bar{p}^h) + j\alpha k(\bar{w}^h, \bar{p}^h)_{\Gamma_h} \\
 & + \sum_{n=1}^{n_{el}} (\mathcal{L}\bar{w}^h, M^e(\mathcal{L}\bar{p}^h - f))_{\Omega_e} \\
 & = (\bar{w}, f) + (\bar{w}, h)_{\Gamma_h}
 \end{aligned} \tag{24}$$

or

$$\begin{aligned}
 & a(\bar{w}^h, \bar{p}^h) + j\alpha k(\bar{w}^h, \bar{p}^h)_{\Gamma_h} \\
 & - \sum_{n=1}^{n_{el}} \int_{\Omega_e} \int_{\Omega_e} \left(\mathcal{L}\bar{w}^h * (\mathbf{x}_0) \cdot g^e(\mathbf{x}, \mathbf{x}_0) \cdot \right. \\
 & \left. (\mathcal{L}\bar{p}^h - f)(\mathbf{x}) \right) d\Omega_x \cdot d\Omega_{x_0} \\
 & = (\bar{w}, f) + (\bar{w}, h)_{\Gamma_h}
 \end{aligned} \tag{25}$$

The method may be viewed as a classical Galerkin finite element method plus an additional term driven by the distributional residual of the coarse scales.

The rather strong assumption that the subgrid scales vanish identically on the boundaries of the element domains has the effect of localizing calculations for the subgrid scales in the sense that the non-local effect of the resolved scales is confined to the element interiors and that the problems are uncoupled from element to element. A Green's function g^e for each element is required and the integral operator M acting on \bar{p}^h couples only the degrees of freedom of \bar{p}^h within an element.

In the numerical implementation, the element Green's function g^e , which is a transcendental function, can be replaced by an approximate polynomial function to facilitate the involved numerical integrations, as is proposed in [76] for the case of finite elements in one dimension and for square elements in two dimensions. It is illustrated in [76] that this subgrid finite element method is superconvergent for one-dimensional Helmholtz problems. For the case of a two-dimensional propagating plane wave problem, however, the accuracy of the method is dependent on the direction of wave propagation. It is therefore believed that the assumption that the fine scales vanish on element boundaries should be removed to improve the accuracy.

Note that by substituting $g^e = \tau \cdot \delta(\mathbf{x} - \mathbf{x}_0)$, where \square is an element parameter, the aforementioned stabilized method, i.e. the Galerkin least-squares method, is recovered and by substituting $g^e = 0$, the standard Galerkin method is obtained.

In the residual-free bubbles method ([77],[78]), the fine-scale equation (19) is solved on individual elements; noting that \bar{p} can be expressed in terms of the coarse scale basis having support in the element in question, a fine-scale basis of residual-free bubbles can be constructed for each element, i.e.

$$\begin{aligned}
 \Pi' \mathcal{N}_a' &= -\Pi'(\mathcal{L}\bar{N}_a - f) \text{ in } \Omega^e \\
 N_a' &= 0 \text{ on } \Gamma^e
 \end{aligned} \tag{26}$$

where $a=1,2,\dots,n_{en}$ is the local numbering of the primary nodes of element e . Thus for each coarse-scale basis function \bar{N}_a , a boundary value problem of type (26) is solved for to obtain the residual-free bubble functions N_a' . This residual-free bubble method provides a generally applicable method, especially on unstructured grids, but involves a

substantial amount of computational effort, especially for Helmholtz problems in multi-dimensional spaces.

An alternative subgrid modelling technique is presented by Cipolla [79]. The coarse scales are expanded in terms of the usual linear element shape functions, while the fine scales are expanded as a series of bubble functions, that are zero on the element boundaries and whose gradient vectors are perpendicular to the gradient vectors of the coarse scale shape functions. By assuming that there is forcing and interelement communication only at the grid scale and not at the subgrid scale, the fine-scale degrees of freedom are condensed out. Due to the aforementioned orthogonality properties of the gradient vectors, the condensation affects only the mass matrix of the coarse scale. The method has been applied for three-dimensional analysis on structured grids and comparable accuracy was obtained for frequencies three times higher than with conventional FEM.

3.5.3. Partition-of-Unity FEM

In the Partition-of-Unity Method (PUM) ([80],[81]), the restriction to regular grids, as is the case for the stabilized methods, is removed. The conceptual idea of this method makes it possible to incorporate analytical knowledge of the exact solution on a general mesh or even using a mesh-free approach of discretization. The PUM is based on an open covering $\{\Omega_i\}$ of the problem domain Ω . With these (compact) support domains Ω_i (often called clouds), some local function spaces V_i , that represent the desired local properties, are associated. In addition, some functions φ_i are locally defined on the supports Ω_i , such that they globally constitute a partition of unity, i.e. their values sum to the unity at each point in the domain ($\sum_i \varphi_i(x) = 1, \forall x \in \Omega$). Then the desired solution space V is simply constructed by multiplying the local approximation spaces with the partition of unity, i.e. $V = \sum_i \varphi_i \cdot V_i$.

In the Partition-of-Unity Finite Element Method (PUFEM) [82] (or often called the Generalized Finite Element Method (GFEM)), a conventional finite element discretization is used as domain covering and the associated Lagrangian finite element shape functions are used as the partition of unity. In this way, the PUM provides a way to construct a conforming element discretization, based on user—specified local approximation spaces that can be chosen very specific to the particular

problem under consideration and that presumably can approximate the exact solution well.

If the local approximation spaces V_i are chosen to be spaces of polynomials, the PUFEM can be understood as a generalization of the classical h-, p- and hp-versions of the FEM. This is illustrated e.g. in [83][84] for solving three-dimensional structural mechanics problems. The number of non-zeros in the resulting stiffness matrix is substantially larger than in the corresponding matrix for the FEM. This is a consequence of the fact that the support of the higher-order shape functions in the PUFEM is larger than the one in the classical FEM. The support of the shape functions is equal to the union of all elements sharing the node associated with the shape function. In the PUFEM, there are only vertex nodes which are, in general, shared by more elements than edge, face and bubble nodes, used in high-order finite elements.

For two-dimensional time-harmonic acoustic problems, governed by the Helmholtz equation, a PUFEM implementation has been proposed recently, in which a set of plane waves, that exactly satisfy the Helmholtz equation, are used as local approximation spaces V_i . In this case, the acoustic pressure approximation takes the form

$$\begin{aligned} p &= \sum_{j=1}^n N_j \cdot \phi_j = \sum_{j=1}^n \sum_{l=1}^m N_j \cdot e^{ik(x \cos \theta_l + y \sin \theta_l)} A_j^l \\ &= \sum_{j=1}^n \sum_{l=1}^m N_j \cdot e^{ik(x \cos \frac{2\pi l}{m} + y \sin \frac{2\pi l}{m})} \cdot A_j^l \end{aligned} \quad (27)$$

with n the number of nodes, m the number of plane wave functions per node and where the conventional polynomial shape functions N_j form a partition of unity. It is not necessary to vary the wave directions uniformly; it is a reasonable thing to do if the actual wave direction is unknown, but if the wave solution has a preferred direction, it would be better a priori to cluster the angles θ_l around that direction. In [85], this plane wave PUFEM approach is used on a square-element discretization for two-dimensional acoustic scattering, while two-dimensional wave diffraction problems on non-square discretizations are studied in [86]. Recently, a similar plane wave PUFEM has been proposed for solving 2D elastodynamic problems ([87]). For such applications, three sets of plane waves (compressional, shear and Rayleigh waves) are being defined per node.

An alternative PUFEM implementation for 2D acoustic problems is proposed in [80],[88] and consists of using generalized harmonic polynomials,

being radial Bessel functions, rather than plane waves as local approximation functions. These functions are also exact solutions of the Helmholtz equation, but a validation on a 2D problem, that has a plane propagating wave in a certain direction as exact solution, indicated that a plane wave PUFEM performs slightly better ([80]).

Since the PUFEM provides a way to include accurate approximation functions into a finite element calculation scheme, it is believed that an enhanced accuracy and computational efficiency could be achieved when solving dynamic problems (in the mid-frequency range). In [82] for instance, a comparison of the PUFEM with the aforementioned GLS and QSFEM and with the conventional FEM has been performed for the two-dimensional Helmholtz problem with a plane propagating wave in a certain direction as exact solution. For this particular case, the PUFEM was superior to all other methods FE methods both in terms of number of degrees of freedom and number of operations (although it must be said that the stiffness matrix in the PUFEM could be created very cheaply due to the particular structure of the structure and the partition of unity).

However, some major difficulties are encountered with the implementation of the PUFEM. Special care must be taken to include essential boundary conditions. Moreover, the numerical integration is computationally much more demanding than for conventional FEM so that the bulk of the computational work is moved from solving the algebraic system of equations to integrating the highly oscillatory integrands, involved with using e.g. plane waves as local approximation spaces. In this respect, special (semi-analytical) quadrature rules are proposed in [89] for the evaluation of these integrals. Another important drawback of the PUFEM is that the resulting system matrices are very ill-conditioned and may disturb the practical convergence of the method.

Although the PUFEM exhibits some promising features as a deterministic mid-frequency modelling approach, there are still many open questions to be answered, especially with respect to its use and robustness for three-dimensional problems with complex geometries.

Note also that in a recent paper [90], the possibilities of the Partition-of-Unity method are being explored in a boundary element context for solving two-dimensional acoustic scattering problems. The method is based on the direct boundary integral formulation of the Helmholtz equation and turns out

to be 3 to 4 times faster than the conventional direct boundary element method.

3.5.4. Discontinuous Enrichment Method

The Discontinuous Enrichment Method (DEM), proposed in [91][92], is based on a finite element discretization of the problem domain. The approximate solution takes the form $p^h = p^P + p^E$, in which p^P is obtained from the standard continuous piecewise polynomial shape functions (similar to the coarse scales in variational multiscale terminology) and p^E is the enrichment field. In contrast with the PUFEM, the enrichment is added to, rather than multiplied by, the polynomial field. The enrichment field may be discontinuous across the element boundaries. Due to this non-conformity of the solution space, inter-element continuity (and Dirichlet boundary conditions) is enforced weakly by suitable Lagrange multipliers.

The weak enforcement of continuity permits the use of free-space solutions as bases for the enrichment. This allows to circumvent the difficulty in attempting to approximate the global fine-scale Green's function of the variational multiscale method and the loss of global effects due to the restriction of residual-free bubbles to vanish on element boundaries. The enrichment field may be eliminated on element level (by static condensation), such that the degrees of freedom consist only of the polynomial field p^P and the Lagrange multipliers.

In a similar way as in PUFEM, the DEM for time-harmonic acoustic problems utilizes an enrichment field based on plane waves. Since the dominant directions of propagation cannot generally be anticipated, a likely implementation of DEM for acoustics is based on an enrichment that is spanned by an even number of plane waves such that for every wave going in one direction there is another one going in the opposite direction. In this case, the complex exponential representations of plane waves can be replaced by real-valued trigonometric functions.

As is the case for PUFEM, the (numerical) integration of terms involving the enrichment functions may require more care than standard FE polynomials. Regularization could be needed to circumvent potential element-level resonance at resolutions below two points per wavelength. If needed, a procedure proposed in [55] for regularizing subdomains can be incorporated in the DEM formulation to regularize elements and thereby overcome this difficulty.

It is shown in [91] for the two-dimensional Helmholtz problem with a propagating plane wave in a certain direction as exact solution that the condition is better than with a comparable PUFEM and that the number of degrees of freedom is 6 to 12 times smaller than with PUFEM.

Note that the use of an enrichment being spanned by free-space solutions that are discontinuous across element boundaries is reminiscent of the T-element approach, that will be discussed later.

3.6. Element-free Galerkin method

Since the generation of a (finite element) mesh of consistent subdomains can be cumbersome and very time-consuming, especially for three-dimensional (acoustic) problems, the possibilities of various meshless techniques are being explored in recent years. These methods still require a set of computational nodes to be sprinkled through the domain, but do not require any pre-specified connectivity of the nodes or locally regular topological structure as is required of traditional meshing.

A particular case of meshless methods is the Element-free Galerkin Method (EFGM) [93]. Its application to acoustic wave propagation problems, governed by the Helmholtz equation, has been studied in [94][95]. The pressure field is approximated as

$$p^h = \sum_{i=1}^n N_i \cdot p_i = \mathbf{N} \cdot \mathbf{p} \quad (28)$$

where \mathbf{p} is a set of nodal values for the pressure field and N_i are the shape functions. These shape functions are constructed using the Moving Least Square method [96], which is defined on a cloud of nodes, that are not connected by elements as it is required for the FEM. The nodes are located at locations x_i inside the problem domain Ω . For each node, a domain of influence is defined, that connects the nodes: two nodes are connected if their domains of influence intersect. For each node, a weight function $w_i(x)$ is defined on its domain of influence and represents the influence of node i at a given position x . This function is unity at the node and decreases when distance to the node increases and is zero outside the domain of influence of the node. In addition, a basis $\mathbf{P}(x)$ of dimension m is defined. Examples are $\mathbf{P}(x)=\{1,x,y\}$, being a linear basis for two-dimensional problems, and $\mathbf{P}(x)=\{1,x,y,x^2,xy,y^2\}$, being a quadratic basis for two-dimensional problems. Using the basis $\mathbf{P}(x)$ and

the weight functions $w_i(x)$, the shape functions are defined as

$$\mathbf{N}(x) = \mathbf{P}^t(x) \mathbf{A}^{-1}(x) \mathbf{B}(x) \quad (29)$$

$$\text{with } \mathbf{A}(x) = \sum_{i=1}^n w_i(x) \mathbf{P}(x_i) \mathbf{P}^t(x_i)$$

$$\text{and } \mathbf{B}(x) = [w_1(x) \mathbf{P}(x_1), \dots, w_n(x) \mathbf{P}(x_n)],$$

where n is the number of nodes. As the weight functions are zero for all nodes that do not influence position x , $\mathbf{A}(x)$ is only the sum of n' matrices, with n' the number of nodes influencing x . The shape functions form a partition of unity, provided that the constant term 1 belongs to the basis $\mathbf{P}(x)$. The pressure field obtained with the EFGM has a higher order of continuity than the solution of FE formulations; if the basis functions P_i ($i=1..m$) $\in C^r(\Omega)$ and the weight functions w_i ($i=1..n$) $\in C^s(\Omega)$, then the MLS approximation $\in C^{\min(r,s)}(\Omega)$. A necessary (but not sufficient) condition for $\mathbf{A}(x)$ not to become singular is that each point of Ω should be located within at least m domains of influence, where m is the size of the basis $\mathbf{P}(x)$.

A shape function associated to a node is not equal to unity at this node, and is not zero at the other nodes, meaning that $p^h(x_i)$ is not equal to the nodal value p_i . This is the reason why Lagrange multipliers are used in the variational formulation of the Helmholtz problem to prescribe the Dirichlet boundary conditions.

For the numerical integrations, involved in the evaluation of the variational formulation using the proposed pressure approximation, an artificial grid independent from the node distribution has to be defined, covering domain Ω as support for Gauss quadrature formulas.

The performance of the EFGM has been compared with conventional and stabilized FE methods in [95] with special focus on the numerical dispersion for two-dimensional problems. It is indicated that the EFGM with a linear basis $\mathbf{P}(x)$ still exhibits numerical dispersion, but less than conventional FEM. When using a frequency-dependent basis $\mathbf{P}^t(x)=\{1,\cos(kx\cos\theta+kysin\theta),\sin(kx\cos\theta+kysin\theta),\cos(-kx\sin\theta+kycos\theta),\sin(-kx\sin\theta+kycos\theta), \dots\}$ with several plane wave directions, the dispersion decreases significantly. However, adding functions to the basis implies increasing the size of the domains of influence, and hence increasing also the number of computations. EFGM implementations using linear polynomial basis and using a basis of 8 plane wave directions exhibit both less dispersion

than a GLS implementation. The numerical dispersion, observed for the EFGM implementation with the wave basis, is similar to the QSFEM. In contrast with most of the stabilized methods, which can only be defined on regular grids, the EFGM doesn't require a regular distribution of nodes, as is illustrated e.g. in [95] for the case of a 2D car-like cavity.

In view of optimizing the computational efficiency of the EFGM, an iterative multilevel meshless method has recently been proposed ([97]). In a first phase, the pressure field is computed with EFGM using a conventional polynomial basis and then a wave basis with $\{1, \cos\theta(x,y,z), \sin\theta(x,y,z)\}$ is used, in which $\theta(x,y,z)$ is the value of the phase of the pressure field at position (x,y,z) , obtained in the previous phase. With this new basis, a second evaluation of $\theta(x,y,z)$ can be done and so on until convergence is reached. The validation on a square 3D cavity with one single plane wave as solution illustrates the potential of this approach.

In this framework, other recent developments include the plans to couple an acoustic EFGM with a PUFEM for Timoshenko beams in order to solve two-dimensional coupled structural-acoustic problems ([98][99]).

3.7 Trefftz approach

The use of polynomial shape functions for field variable expansion, as in the finite element method, has some major drawbacks. The polynomial functions are no exact solutions of the governing partial differential equations. Moreover, the prediction accuracy for derived secondary variables is smaller than for the primary field variables, since the order of the polynomial expansions for the (higher-order) derivatives is lower than the polynomial order of the primary field variable expansions. These drawbacks would vanish, if the solutions for the field variables are expressed in terms of exact solution functions of the governing partial differential equations. In 1926, Trefftz [100] utilised already this idea and proposed an alternative for the Rayleigh-Ritz method. Almost half a century later, researchers became interested again in Trefftz' ideas, when their potential advantages for an efficient use in numerical prediction techniques were recognised. This has led to a modelling method, which is commonly denoted as *Trefftz method*. The *spectral finite element method* combines the Trefftz approach with the element approach of the finite element method, but its use is restricted to continuum problems with some particular geometrical shapes. The *T-element*

method combines both approaches in a more generally applicable procedure.

This section describes the basic concepts of these three methods and indicates their current limitations, which explain why, despite the promising features regarding computational efficiency, these methods still require a substantial amount of research, before they may become mature and versatile modelling alternatives.

3.7.1. Trefftz method

The original ideas, proposed by Trefftz ([100]), have led to the development of the indirect Trefftz method. More recently, these ideas have served also as the basis for an alternative modelling approach, commonly denoted as the direct Trefftz method. A more detailed overview of the Trefftz method can be found in [101].

The main feature of the *indirect Trefftz method* is the approximation of the solution in terms of functions, which belong to a complete set of functions that satisfy exactly the governing (homogeneous) equations. In the original method, proposed by Trefftz, a variational formulation was used, in that the contributions of the functions to the solution result from a linear system of equations, imposing a stationarity condition on an energy functional. Since the proposed solution exhibits only some approximation errors on the boundary conditions, only some boundary integrals are involved in the evaluation of the energy functional for the proposed solution. In this way, the method may offer some computational benefits over the Rayleigh-Ritz method, in which also some domain integrals are involved in the functional evaluation. Although the original indirect Trefftz method utilises a variational formulation, the contributions in the proposed solution can also be obtained from the approximation of the boundary conditions in a collocational, a least-squares or a weighted residual scheme. The latter yields the same system of equations as the originally proposed variational formulation, when the weighting functions are selected from the same complete function set as used for the solution.

The completeness of the function set indicates the ability to represent any possible solution in a given continuum domain. In honour of the originator of the method, the name of T(Trefftz)-complete functions is adopted for such sets. Important contributions to the development of a mathematical basis for the construction of such T-complete function sets are made by Ismael Herrera and his

collaborators. They presented T-complete sets for two- and three-dimensional problems of Laplace and Helmholtz equations ([102]), biharmonic equations ([103]) and Stokes problems ([104]).

The *direct Trefftz method* is based on a weighted residual formulation of the governing equations, using T-complete functions as weighting functions. For this particular weighting function selection and by applying integration by parts, the weighted residual formulation consists only of boundary domain integrals. As in the boundary element method, the boundary surface is discretized into small boundary elements and the solution approximations for the field variables, occurring in the boundary domain integrals, are expressed in terms of classical (polynomial) shape functions, defined within each boundary element. The shape function contributions result from the discretized version of the weighted residual, now boundary integral, formulation, combined with the boundary conditions of the considered problem. The term 'direct' indicates that the unknowns in the system of equations are physical quantities on the boundary surface of the considered continuum domain. This formulation was firstly presented by Cheung *et al.* ([105]) in the two-dimensional potential problem and extended to the two-dimensional elastic problem ([106]), the two-dimensional Helmholtz problem ([107]) and the plate bending problem ([108]).

The main asset of the indirect Trefftz method is the fact that an approximation error is only induced in the representation of the boundary conditions. As a result, a high prediction accuracy can be obtained with significantly smaller models than with corresponding element based models, which are based on shape function expansions that are no exact solutions of the governing equations. Therefore, a Trefftz approach exhibits a much higher potential for getting significant reductions in computational load. However, a disadvantage is that, in order to obtain convergence, a complete set of globally defined shape functions is required, i.e. a set of exact solution functions, which is able to represent any possible field variable distribution in the considered continuum domain. For various types of continuum problems, complete shape function sets have already been defined, for which the theoretical convergence is proven. However, their practical convergence is seriously disturbed or even impossible, due to the poor condition of the involved model matrices. This explains why the indirect Trefftz method has not yet become a generally applicable modelling alternative for the element based prediction techniques.

Only for some specific dynamic applications, there are some successful research activities that employ the Trefftz-idea of expressing the numerical solution in terms of functions, that exactly satisfy the governing equations. One of them is the development of the source simulation technique. The basic idea of the method is to calculate the acoustic radiation or scattering of complex-shaped structures by replacing the vibrating structure with a distribution of multipoles, called sources, located inside the structure, such that the acoustic field radiated by these sources verifies the boundary conditions on the structure ([109],[110]). The source distribution results from a collocational or a weighted residual formulation of the boundary conditions. Various formulations and implementations have been proposed ([111][112][113][114][115]). The method can be an efficient modelling alternative, if the acoustic radiation of the structure can be approximated with a sufficiently low number of sources. Although there are some differences between the various formulations, they all share, however, the property that the resulting system matrices are ill-conditioned, which can jeopardize the convergence and robustness of the method, and that the performance of the method is strongly dependent on the choice of the location and the number of sources. A similar idea may be used also to calculate interior acoustic fields ([116],[117]). Recently, an energy source simulation method [118] is proposed. A similar approach is used where energy density sources are used instead of pressure sources in an effort to produce the prescribed boundary intensity instead of the normal velocity. In the non-dimensional dynamic influence function method (NDIF), a distribution of sources is defined on the boundary of the structure. The source strengths result from a collocational formulation of the boundary conditions. Regular Bessel functions of the first kind and zero order used as source functions for the expansion of the pressure field in two-dimensional acoustic cavities ([119]) and for the displacement fields in membranes ([120],[121]), while regular Bessel functions of the first and second kind and zero order are used as regular source functions for the expansion of the normal displacement in plates ([122]).

Another indirect Trefftz implementation is proposed in [123] for the free vibration analysis of polygonal membranes and in [124] for the free-vibration analysis of two-dimensional polygonal acoustic cavities. The polygonal problem domain is considered as a collection of edges of semi-infinite planes and waveguide-type base functions are used

for the field variable expansions. Each waveguide-type base function varies trigonometrically in the direction parallel to one of the edges with periodicity equal to the edge length, while it takes a wave propagating form in the direction, perpendicular to that edge, such that the base function exactly satisfies the governing Helmholtz equation. The contributions of these base functions to the dynamic response result from a collocational formulation of the boundary conditions.

3.7.2. spectral FEM

The spectral finite element method combines the Trefftz approach, in which the field variable solutions are expressed in terms of exact solution functions of the governing dynamic equations, with the element concept, in which some field variable-related quantities, specified at some nodes on the element boundary, are related to some generalised excitation forces at these nodes. This subsection briefly describes the basic principles. For a more detailed description, the reader is referred to e.g. [125],[126].

For several continuum problems, of which the steady-state dynamic behaviour is governed by ordinary differential equations in one independent spatial variable, the continuum domain may be discretized into a number of one-dimensional elements, such that all geometrical and material discontinuities and external excitations occur only at the interfaces between the elements. In this way, the field variables within each element are governed by homogeneous ordinary differential equations and their exact solutions may be expressed in terms of the finite number of homogeneous solutions of these equations. By constructing appropriate linear combinations of these homogeneous solutions, nodal shape functions can be obtained. Each of them represents an exact solution, which has a unit value for one degree of freedom in one of the two endpoints of the element and in which all other endpoint degrees of freedom are zero. Based on the shape function expansions for the field variables within an element, a dynamic stiffness matrix can be constructed, which exactly relates the nodal field variables of the element to the nodal excitations, applied at the endpoints of the element. The dynamic stiffness matrices of the different elements are then assembled by enforcing the boundary conditions of the considered problem and by enforcing the compatibility of the nodal field variables and the nodal and external excitations at the element interfaces. The obtained global matrix equation may then be solved for the unknown nodal

field variables and the back-substitution of these results into the shape function expansions yields the exact solution for the considered continuum problem. Finnveden ([127]) illustrated, for example, how the element shape functions are obtained for the dynamic displacements in a structural beam element and applied the dynamic stiffness method to calculate the steady-state dynamic response in a railway car frame structure, which consists of several structural beams. Achmida and Arruda ([128]) applied the method to model three-dimensional frame structures. When a distributed external excitation is applied to the continuum domain, the ordinary differential equations that govern the steady-state dynamic variables within the elements are inhomogeneous for at least one element of the domain. Although the homogeneous nodal shape functions in these elements are no exact local solutions anymore, they still offer a beneficial convergence rate compared to the polynomial shape functions, used in a conventional finite element discretization. Lee and Lee ([129]) illustrated this, for example, for the case of a structural beam with a distributed dynamic force load.

The dynamic behaviour of general two- and three-dimensional continuum problems is governed by partial differential equations in one time-variable and, respectively, two and three independent spatial variables. Provided that the problem geometry, material properties and boundary conditions are piecewise constant in one spatial variable, say the x -coordinate, the spectral finite element method may be applied for the prediction of the dynamic behaviour in each piecewise constant part of the continuum domain. Typical applications include built-up structures that consist of waveguide-type components such as extruded profiles. All calculations are performed in the frequency domain; time-domain responses may be obtained by performing an inverse FFT procedure on the frequency results. The field variable distributions in the cross-sectional plane of the continuum, i.e the y -plane in a two-dimensional continuum or the yz -plane in a three-dimensional continuum, are expressed in terms of conventional finite element shape functions. The cross-sectional nodal degrees of freedom are then only function of the spatial coordinate x and are governed by a set of ordinary differential equations. By assuming that all cross-sectional degrees of freedom have a harmonic x -dependence with a certain common wavenumber, the set of ordinary differential equations and boundary conditions turn into an eigenvalue problem, that can be solved for each frequency of interest. This results in an infinite set of (frequency-

dependent) possible wavenumbers and associated cross-sectional eigenmodes. A truncated set of these eigensolutions can then be used to expand the field variable distributions and to construct a subsequent dynamic stiffness matrix for each piecewise constant part of the considered continuum domain. The various dynamic stiffness matrices are then assembled into one global spectral FE model. In this way, the spectral FE method is a powerful tool for wave propagation problems, in that the method exactly models the inertia properties and therefore the elements can be large, in fact spanning the region between discontinuities. Furthermore, the effects of damping, visco-elasticity and higher order structural models can easily be incorporated. Its main restriction relates to the geometrical complexity of the structure. Till now, the main applications include the dynamic analysis of rib-stiffened plates ([130]) and (fluid-filled) cylindrical pipes ([131][132][133]).

3.7.3. T-elements

The numerical condition problems, involved with the practical implementation of the indirect Trefftz method, may be circumvented, to some extent, by dividing the continuum domain into smaller subdomains. This phenomenon has led to the construction of T(Trefftz)-elements ([134],[135]), which allow the introduction of the Trefftz approach into a standard finite element scheme and which can be used, in contrast with the spectral finite element method, for continuum problems with geometrical shapes of any kind.

In the conventional finite element method, the field variables are expanded in terms of shape functions, which are locally defined within the elements. The conformity of the element discretization is verified a priori and the 'equilibrium' and constitutive relations, expressed by the governing equations, are then restored approximately in an average integral sense. An alternative approach, based on the Trefftz methodology, proceeds in exactly the opposite way. The field variables are expanded in terms of T-complete functions, which satisfy a priori the governing equations within each element, while the interelement continuity and the boundary conditions are enforced in an average integral sense.

Two types of T-element approaches can be identified, i.e hybrid-Trefftz elements and least-squares Trefftz elements. *Hybrid-Trefftz elements* combine the Trefftz methodology with the concept of hybrid finite element modelling. Two types of degrees of freedom are involved in the definition of hybrid-Trefftz elements. The first type are the

contributions of the T-complete functions to the field variable distributions within the elements. The second type results from the introduction of auxiliary 'frame'-functions. These functions are only defined on the element boundaries and are expressed in terms of conventional (polynomial) shape functions and associated nodal frame degrees of freedom. The element formulation results from enforcing the conformity of the frame functions and the corresponding T-complete function expansions for the field variable distributions on the element boundaries, as well as enforcing the interelement conformity and the external boundary conditions in an average integral sense. The subsequent relation between the nodal frame degrees of freedom of an element and the contributions of the T-complete functions allows the elimination of the latter degrees of freedom in the element formulation. This yields, in exactly the same way as for conventional finite element formulations, symmetric element matrices, which relate the nodal frame degrees of freedom to the nodal 'forces'. Therefore, the linking of hybrid-Trefftz elements can be accomplished via a standard element assemblage procedure and T-element discretizations may also be coupled with conventional finite element discretizations.

The *least-squares Trefftz element* formulation is based only on the contributions of the T-complete functions to the field variable distributions within the element, without the introduction of auxiliary frame degrees of freedom, but where the interelement conformity and the boundary conditions are enforced directly in a least-squares sense.

In comparison with conventional finite elements, Trefftz-elements have some beneficial properties.

- Since the T-complete functions satisfy a priori the governing equations, a substantially improved accuracy is (often) obtained for a given number of degrees of freedom.
- The element formulation offers a greater flexibility in choosing the element geometry, since the integrations, involved in the construction of the element matrices, are confined to the element boundaries.
- The element formulation is suited for the development of an adaptive p-method approach, based on very simple error estimators.
- Difficult singularity problems and concentrated or discontinuous loads may efficiently be handled without troublesome local mesh refinements by simply adding some special-purpose functions to the solution expansion.

Some promising implementations of the T-element approach have been obtained recently for acoustic

([136][137][138][139]) and for elastodynamic ([140]) analysis. Similar ideas are adopted in the current developments of the Variational Theory of Complex Rays (VTCR) for the mid-frequency analysis of lightly damped elastic structures ([141][142][143][144]). In the latter method, a structure is considered as an assembly of several substructures. A superelement is associated with each substructure and its degrees of freedom correspond to local basic modes. These local basic modes are complex rays, which are exact solutions of the governing equations and which can be divided into families related to interior, edge and corner zones. The solution is supposed to be well described locally in the vicinity of a position on the structure as a superposition of an infinite number of these local basic modes. The solution expansion is defined on two scales, in that the local modes are defined explicitly on a short wavelength scale, while the unknowns, being the contributions of the local modes, are slowly varying, large-wavelength quantities. An appropriate variational formulation of the problem enables to verify, on the average, the boundary and transmission conditions. The formulation allows approximations, which are a priori independent within substructures. These approximations do not necessarily need to verify the transmission conditions a priori. In practice, the variational formulation associates each superelement with, on the one hand, an elementary matrix which represents the interaction of the fields with one another and on the other hand a right-hand side which represents the interaction between the fields and the boundary conditions. The transmission conditions at the interface are taken into account automatically at the assembly stage. The VTCR method can be regarded as a 'true' mid-frequency technique, in that an information filter is used: only effective quantities, which are related to e.g. elastic energy and dissipation work, are retained from the computed slowly varying, large-wavelength mode contributions.

In its current stage of development, the T-element method suffers, however, from a severe limitation. The efficiency of the currently available hybrid-Trefftz and least-squares Trefftz element formulations in fulfilling the boundary conditions and element conformity is strongly dependent on the considered problem. Hence, for the T-element method to become an attractive alternative for the conventional finite element method, there is a need for some alternative element formulations, whose efficiency is less problem-sensitive.

4. Wave based prediction technique

None of the above described new developments, at least not in their current stage of development, are capable of combining three important properties, i.e. an enhanced computational efficiency, together with a general applicability with respect to geometrical complexity and the ability to account for mutual fluid-structure coupling effects in coupled vibro-acoustic systems. However, a novel wave based prediction technique ([145][146][147]), which is being developed at the KULeuven - Noise and Vibration Research group and which adopts an indirect Trefftz approach, exhibits some promising features with respect to these three properties.

In contrast with the element based techniques, the structural and acoustic domains are no longer divided into small elements. The dynamic variables in the entire domains, or at least in large subdomains, are expressed in terms of wave functions, which are exact solutions of the governing dynamic equations. The contributions of the wave functions to the coupled dynamic response result from a weighted residual or least-squares formulation of the structural and acoustic boundary conditions. An important breakthrough of the novel prediction technique is the definition of complete wave function sets, which yield coupled vibro-acoustic prediction models, whose poor numerical condition no longer prevents the prediction results from converging towards the exact solution. For the steady-state pressure field in two-dimensional acoustic cavities, for instance, the following wave function expansion is proposed

$$p = \sum_{m=0}^M p_m \cdot \cos\left(\frac{m\pi}{L_x} x\right) \cdot e^{\pm j \sqrt{k^2 - \left(\frac{m\pi}{L_x}\right)^2} y} + \sum_{n=0}^N p_n \cdot e^{\pm j \sqrt{k^2 - \left(\frac{n\pi}{L_y}\right)^2} x} \cdot \cos\left(\frac{n\pi}{L_y} y\right) \quad (30)$$

in which L_x and L_y are the dimensions of the (smallest) rectangular domain that encloses the actual problem domain. p_m and p_n are the unknown wave function contributions. It is proven in [145] that the convexity of the problem domain is a sufficient condition for the solution expansions (30) to converge, in the limit for $M, N: 0 \rightarrow \infty$, towards the exact response. Although a firm mathematical proof is still under construction, the author has not yet

found an example, counterproving the statement that any type of non-convex domain can be decomposed into some subdomains and that a convergent expansion for each subdomain is obtained by using a wave function selection of type (30), in which L_x and L_y are the dimensions of a rectangular domain, enclosing the subdomain. Note that, when the problem domain must be decomposed for convergence reasons, some additional boundary conditions must be included to ensure the continuity of the pressure and normal fluid velocity along the common interfaces between the various subdomains.

These modelling concepts and the definition of complete wave function sets, based on the dimensions of an enclosing domain, have already been successfully applied for various types of coupled vibro-acoustic problems. In its current stage of development, the proposed wave based prediction technique enables already the accurate prediction of the following steady-state dynamic field variable distributions in coupled vibro-acoustic systems:

- pressure fields in two-dimensional bounded acoustic domains with arbitrary geometries ([146],[149]),
- pressure fields in three-dimensional bounded acoustic domains with arbitrary geometries ([147],[148]),
- pressure fields in two-dimensional unbounded acoustic domains ([150],[151],[152]),
- in-plane and normal displacement fields in one- and two-dimensional flat plates with arbitrary geometries ([145]),
- displacement fields in one-dimensional curved structures, which may be regarded as assemblages of cylindrical shell sections ([153]),
- in-plane and normal displacement fields in coupled flat plates with arbitrary geometries ([154]),
- dynamic field variable distributions in two-dimensional fluid-saturated, poro-elastic domains with arbitrary geometries ([145]).

It is illustrated through several validation examples that, for all these types of field variables, the proposed wave based prediction technique yields accurate numerical prediction results, which converge towards the exact solutions.

The new prediction technique has an important contribution to the advance of coupled vibro-acoustic modelling, due to its enhanced computational efficiency, compared with the existing prediction methods. In comparison with the finite element method, for instance, the wave based

prediction technique has some beneficial (+) and some disadvantageous (-) properties regarding computational efficiency.

- (+) Coupled vibro-acoustic wave models are substantially smaller than corresponding finite element models. This is due to the fact that the proposed field variable expansions satisfy a priori the governing dynamic equations, so that approximation errors are only involved with the representation of the boundary conditions.
- (+) The expansions for derived secondary field variables, such as fluid velocity and structural stress, have the same spatial variation as the primary field variables, i.e. the acoustic pressure and structural displacement. This is advantageous for the convergence rate, especially in the case of coupled vibro-acoustic problems, for which the effect of the fluid on the structure is pressure-controlled, but for which the effect of the structure on the fluid is velocity-controlled.
- (-) Since the wave functions are complex functions, defined within the entire continuum domain or at least within large subdomains, the wave model matrices are fully populated and complex.
- (-) Due to the implicit frequency dependence of the wave functions, the wave model matrices cannot be decomposed into frequency independent submatrices, which implies that a wave model must be recalculated for each frequency of interest.
- (-) Due to the global and oscillatory nature of the wave functions and due to the poor condition of a wave model, the involved numerical integrations are computationally more demanding.

It is identified that the effects of the beneficial properties on the computational efficiency clearly predominate the effects of the disadvantageous properties. The accuracy and associated computational efforts, involved with wave models and corresponding element based models, have been compared for several (coupled vibro-)acoustic validation examples. These comparisons have revealed the beneficial convergence rate of the proposed wave based prediction technique, in that it provides highly accurate prediction results with a substantially smaller computational effort than the finite element method. Due to these beneficial convergence properties, the practical frequency limitation of the wave approach is significantly larger than for the element based techniques. Therefore, the wave modelling concept offers an adequate way to comply with the challenge of

extending the applicability of deterministic prediction techniques towards higher frequencies.

In a next step, the research work will aim at the extension of the promising wave modelling concept towards a computationally efficient, generally applicable and easily accessible technique, which provides accurate (coupled) vibro-acoustic predictions in the low- and mid-frequency range. To achieve these objectives, the research efforts will mainly focus on the following tasks.

- *extension of the applicability*

In view of extending the applicability of the wave approach, the definition of complete wave function sets for the description of the dynamic displacements in three-dimensional curved structures will be explored. In addition, the approach, which has already been successfully applied for the modelling of the vibro-acoustic coupling interaction between the solid and fluid phase in two-dimensional poro-elastic sound insulation materials, will be extended for the use of various three-dimensional poro-elastic material models. Finally, it will be investigated how the wave approach can be used for pressure field predictions in three-dimensional unbounded acoustic domains.

- *automation of the modelling concept*

For the technique to be easily accessible without full knowledge of the mathematical background, its implementation must exhibit a substantial level of automation. In a similar way as the definition of parent elements, used in conventional element based techniques, the possibility will be explored of defining a standard formulation for each component type that may occur in a vibro-acoustic system (acoustic, structural and poro-elastic 'wave elements'). In this way, the wave model for each component may then be automatically generated from the problem dependent specification of the geometrical and material properties of each component and the global wave model may then be obtained from an automated assembly of the component models, based on the boundary conditions and external excitation(s) of the considered vibro-acoustic problem.

- *coupling with element based models*

In the finite element method, a system with a complex geometrical shape is transformed into a large collection of small, geometrically simple entities. This element discretization forms the

base for the wide applicability of the finite element method, however, at the expense of huge computational efforts. The strength of the computationally efficient wave modelling concept originates mainly from the fact that large regions in a continuum domain, which are governed by the same dynamic equations, are no longer discretized. Therefore, a mixed modelling approach will be adopted, in which the strengths of both methods are optimally merged by using a wave approach for modelling the large, geometrically simple subregions in a (coupled) vibro-acoustic system, and a finite element approach for modelling the local, geometrically complex details. In this framework, some preliminary, but promising results have already been reported in [155],[156].

5. Conclusion

In recent years, a vast amount of research has been initiated in a quest for an adequate prediction technique for the mid-frequency analysis of acoustic and (coupled) vibro-acoustic systems. In the mid-frequency range, the computational efforts of conventional element based techniques such as FEM and BEM become prohibitively large, while the basic assumptions of the probabilistic techniques such as SEA are not yet valid.

This paper has discussed the various methodologies that are currently being explored in this perspective. The main focus of the paper lies on the methodology that looks for deterministic techniques with an enhanced convergence rate and computational efficiency compared to the conventional element based methods in order to shift the practical frequency limitation towards the mid-frequency range. The discussion includes the development of efficient solvers for large FE models, the use of adaptive strategies to optimize the mesh density in view of uniform error distribution, the use of domain decomposition techniques to reduce the model sizes or to subdivide the original large problem into many smaller subproblems, the use of knowledge-based FE methods that try to include a priori knowledge about the solution into the FE implementation, the use of meshless techniques to reduce the efforts involved with mesh generation, and the use of the Trefftz approach, in which the solution expansions exactly satisfy the governing dynamic equations. None of these new developments, at least not in their current stage of development, are capable of combining three important properties, i.e. an enhanced computational efficiency, together with a

general applicability with respect to geometrical complexity and the ability to account for mutual fluid-structure coupling effects in coupled vibro-acoustic systems. However, a novel wave based prediction technique, which is being developed at the KULeuven - Noise and Vibration Research group and which adopts an indirect Trefftz approach, exhibits some promising features to comply with the mid-frequency modelling challenge. Various recent validations have revealed that the method can be applied for coupled vibro-acoustic systems of moderate geometrical complexity and that it has a significantly enhanced convergence rate. In view of obtaining a generally applicable, highly efficient coupled vibro-acoustic prediction technique, the next research steps will mainly focus on the extension of its applicability to systems of arbitrary geometrical complexity.

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