Frequency domain, parametric estimation of the evolution of the time-varying dynamics of periodically time-varying systems from noisy input-output observations

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Abstract
This paper presents a parametric, frequency domain identification method for modeling continuous- (discrete-) time, periodically time-varying systems from input-output measurements. In this framework both the output as well as the input are allowed to be corrupted by stationary noise (= errors-in-variables approach). Furthermore, it is assumed that the system under consideration can be excited by a broad-band periodic signal with a user-defined amplitude spectrum (i.e. multisine), and that the periodicity of the excitation signal, \( T_{\text{exc}} \), can be synchronized with the periodicity of the time-variation, \( T_{\text{sys}} \), (i.e. \( T_{\text{exc}} \| T_{\text{sys}} \in \mathbb{Q} \)). Under these conditions the system can reach a steady state. Besides, two different concepts of a transfer function for time-varying systems (called the frozen transfer function and the instantaneous transfer function) are also introduced. A clear distinction between both is made, and either can be estimated with the proposed identification scheme. It is up to the users to decide which definition suits best their purpose. Uncertainty bounds on all/most frozen model-related quantity (such as frozen transfer function, frozen poles, frozen resonance frequency, ...) are provided in this paper as well. Finally, the identification algorithm is demonstrated on an extendible robot arm.

1 Introduction

The linear time-invariant (LTI) identification techniques are well-established nowadays [8, 14]. However, there are situations where the time-invariant assumption is not met. This happens in real-life systems where the dynamics are changing over time [4]. The causes of the time-variation in the system can be catagorized as follows:

- The time-varying character is inherently present in the system and is dictated by the random nature of the system, [5, 17] (e.g., pitting corrosion in metals, vibration of a helicopter rotor, (bio-) chemical processes, etc.).

- The time-varying behavior is artificially created by an external parameter, the so-called scheduling parameter, [6] (e.g., extendible robot arm, flight flutter, electronic circuit whose properties depend on an external voltage source, etc.). This kind of system fits into the framework of linear parameter varying (LPV) systems, where the dynamics of the system depend on the external (scheduling) parameter. When the scheduling parameter follows a certain trajectory over time, the systems’ dynamic becomes time-varying to some extent.

- When a nonlinear time-invariant system is linearized around the periodic (stable) orbit of the nonlinear system, the linearized system will exhibit a time-varying behavior in the neighborhood of the stable orbit, [18] (e.g., mechanical systems with a nonlinear stiffness, power distribution networks, ... ).
Many physical structures, in particular mechanical structures, are described by partial differential equations (structural analysis, modal analysis, heat transfer, fluid mechanics, electromagnetism, ...), where it is a common practice to use finite element methods (FEM) to solve the problem. The FEM gives rise to a set of ordinary differential equations (ODE), where each matrix element determines the physical relationship between the applied inputs at some points and the resulting outputs at others. Therefore, it can be modeled by an ODE at a certain point in the structure (described as a transfer function of a Single-Input Single Output (SISO) system). This framework of lumped systems, simplifies the analysis of the system under consideration and the corresponding identification methods. Extending this idea to systems (structures) with a time-varying behavior shows that it is meaningful to describe time-varying systems with time-varying ODEs.

In this article, we only focus on periodically time-varying (PTV) systems where it is assumed that the PTV system can be well-described by a SISO ODE with periodically time-varying parameters. PTV systems have been covering a wide range of applications, especially in the field of mechanics where many systems that sustain a periodic motion (e.g., gear boxes, electrical motors, fans, shafts, helicopter blades, ...) show a PTV behavior [3].

Numerous parametric identification methods for linear periodically time-varying (LPTV) systems in different fields of engineering have been described in the literature. An overview of some existing parametric identification methods for LPTV systems is hereunder given. Using the lifting and the Fourier series expansion technique, [1] presented a Multi-Input Multi-Output (MIMO) LTI method to identify continuous state-space LPTV systems from the free response of mechanical systems. This identification technique is applied in [18] to identify nonlinear time-invariant (NLTI) systems giving a 2-step procedure. In addition, in [2] a frequency domain identification scheme is developed to identify the behavior of the blades of wind turbines from output-only measurements. The modal parameters (i.e. Floquet exponents and the corresponding mode shapes) are extracted from the output power spectrum using classical LTI techniques (e.g. the peak-picking algorithm). Recently, an identification method in [19], making use of Daubechies wavelets within a state space approach, is elaborated to estimate the time-varying system matrix $A(t)$ and the corresponding eigenvalues. In the digital processing world, discrete-time LPTV systems are more suited [11], where [20] proposed a frequency domain estimate of the alias components of LPTV systems using a finite impulse response (FIR) approach.

Many identification methods that have been developed in the literature (such as the ones mentioned above) start from a control perspective: the input is known and the output is disturbed by noise — this is known as the generalized output-error framework. In this work the more general case will be tackled where the measurements of both the input and the output are noisy — we are working in an errors-in-variables environment. In addition, the modeling is done in the frequency domain which has the advantages that: i) the errors-in-variables approach is as simple as the generalized output-error problem; ii) both the continuous-time and the discrete-time cases can be handled with the same identification algorithm; iii) a nonparametric weighting of the noise variance is readily incorporated into the cost function (see further on), providing consistent estimates; iv) it is straightforward to select the frequency band of interest. Finally, we also provide practical uncertainty bounds of any frozen model-related quantity (such as the frozen transfer function, frozen poles, frozen resonance frequency, ...), which can be used to assess the quality of the obtained estimates.

The paper is organized as follows. In Section 2 the considered model class used for the identification is discussed. The definition and the motivation of the frozen system and its related model quantities (e.g. frozen transfer function, frozen modal parameters, ...) are also given in this section. Within this section two concepts of a tranfer function are introduced and distinguished: the frozen transfer function and the instantaneous transfer function. Section 3 describes the identification procedure to set up a consistent estimator. While in Section 4 it is shown how to compute uncertainties on the frozen model quantities in order to build uncertainty bounds. The methodology is then applied to real measurements in Section 5. Finally, in Section 6 we draw some general conclusions.
2 System description

2.1 System model

In this work the focus will be put on single-input single-output (SISO) periodically time-varying (PTV) systems that can be well-described by an ordinary differential equation (ODE)

\[ \sum_{n=0}^{n_{a}} a_n(t) \frac{d^n y_0(t)}{dt^n} = \sum_{n=0}^{n_{b}} b_n(t) \frac{d^n u_0(t)}{dt^n} \]

(1)

with periodically time-dependent system parameters \( a_n(t) = a_n(t + T_{sys}) \) and \( b_n(t) = b_n(t + T_{sys}) \) with \( T_{sys} \) the periodicity of the time-variation and where \( \{u_0(t), y_0(t)\} \in \mathbb{R} \) stand for, respectively, the undisturbed input and output signals. Furthermore, it is assumed that the system parameters, \( a_n(t) \) and \( b_n(t) \) are band-limited (i.e. the bandwidth of the power spectra of \( a_n(t) \) and \( b_n(t) \) must be much smaller than the Nyquist frequency). Note that the model orders (i.e. \( n_a \) and \( n_b \)) in the model equation are time-invariant. Hence, it is assumed in this paper that the model complexity will not change during the measurement process. This assumption will allow us to determine the orders \( n_a \) and \( n_b \) from a time-invariant (TI) experiment by using well-known LTI-techniques, such that, when performing a PTV experiment, the obtained orders can be used for the PTV model.

Since by definition the system parameters \( a_n(t) \) and \( b_n(t) \) are time-periodic, and band-limited as assumed, they are equal to a truncated Fourier series, viz.

\[ a_n(t) = \sum_{k=-n_{a}}^{n_{a}} A_{[n,k]} e^{jk\omega_{sys}t}, \quad b_n(t) = \sum_{k=-n_{b}}^{n_{b}} B_{[n,k]} e^{jk\omega_{sys}t} \]

(2)

with \( \omega_{sys} = 2\pi f_{sys} = \frac{2\pi}{T_{sys}} \) the base (angular) frequency of the time-variation and \( X_{[n,k]} \) the \( k^{th} \) Fourier coefficient of \( x_n(t) \) \((x = \{a, b\} \) and \( X = \{A, B\}\)). The degree of time-variation is thus determined by the length of the Fourier decomposition (i.e. the truncation order \( n_{a} \) and \( n_{b} \) in (2)).

As mentioned in the introduction, the estimation will be carried out in the frequency domain. Before doing the transformation, important assumptions on the excitation signal, \( u_0(t) \), will be formulated such that the frequency domain model becomes as simple as possible for the identification. The true input \( u_0(t) \), sampled at the rate \( f_s = 1/T_s \) and observed in the time span \([0, T]\), is periodic and band-limited

\[ u_0(nT_s) = \frac{1}{\sqrt{N_{exc}}} \sum_{k_{exc} \in \mathbb{R}_{exc}} U_k e^{j(k2\pi f_{exc}n + \varphi_k)}, \]

(3)

where \( f_{exc} = \frac{1}{f_{exc}} \) stands for the base frequency of the multisine, \( U_k \geq 0 \) is user defined, \( \varphi_k \) are randomly chosen phases s.t. \( \mathbb{E} \{ e^{j\varphi_k} \} = 0 \) and with \( \mathbb{R}_{exc} \) the set of excited frequencies (i.e. a subset of integer numbers). \( N_{exc} = \#\mathbb{R}_{exc} \) is a scaling factor to ensure that the power of the excitation signal is independent of the number of excitation lines (\( \#\mathbb{R}_{exc} \) stands for the number of elements in the set \( \mathbb{R}_{exc} \)). This kind of excitation signals appears to be very useful in practice. For instance, it allows us for a clear, nonparametric distinction between the system noise and the nonlinear distortions if the system behaves to some extent nonlinearly, [10]. Multisines also provides an estimate of the nonparametric model of the (co-)variances of the colored input-output noise from successive periods of the measured input-output signals (see Section 3.3). In addition, it is supposed that an integer number of periods of the input signal and the time-variation are observed, viz.

\[ T = qT_{exc} = pT_{sys} = NT_s, \quad \{q,p,N\} \in \mathbb{N}_0. \]

(4)

Under this circumstances the true output in (1) will be periodic in steady state, \( y_0(t + T) = y_0(t), \) such that no transient term pops up when transforming (1) to the frequency domain.
Starting from the periodicity condition on the input-output signals (i.e. \( u_0(t+T) = u_0(t) \), \( y_0(t+T) = y_0(t) \)), and based on the fact that in steady state conditions the continuous Fourier transform of the signals at \( \omega_k = 2\pi \frac{k}{T} \) can exactly be replaced by the discrete Fourier transform (DFT) at that frequency \( \omega_k \)

\[
X(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(nT_s)e^{-j2\pi k n/N},
\]

it can then be proven that the relation between the true input-output DFT spectra becomes

\[
\sum_{n=0}^{n_a} \sum_{k=-n_{ka}}^{n_{ka}} A[n,k] \left( j \frac{2\pi}{T} (l - pk) \right)^n Y_0(l - pk) = \sum_{n=0}^{n_b} \sum_{k=-n_{kb}}^{n_{kb}} B[n,k] \left( j \frac{2\pi}{T} (l - pk) \right)^n U_0(l - pk),
\]

with \( p = \frac{T}{T_{sys}} \). Note that for \( n_{ka} = n_{kb} = 0 \), model (6) boils down to the well-known LTI model in the frequency domain, [14]. Evaluating (6) at the (shifted) DFT bins \( l = \{-\frac{N}{2}, -\frac{N}{2}-1, ..., 0, ..., \frac{N}{2}-2, \frac{N}{2} \} \) if \( N \) is even or at the bins \( l = \{-\frac{N-1}{2}, -\frac{N-1}{2} - 1, ..., 0, ..., \frac{N-1}{2} - 1, \frac{N-1}{2} \} \) if \( N \) is odd, and grouping the equations together into a single matrix equation, we obtain in either case as frequency domain model

\[
\mathbf{A}(\theta_x)\mathbf{Y}_0 = \mathbf{B}(\theta_b)\mathbf{U}_0
\]

where \( \mathbf{X}_0 = \text{DFT}\{x_0\} \in \mathbb{C}^{N \times 1} \), \( \mathbf{x}_0 = [x_0(0) x_0(T_s) x_0(2T_s) ... x_0((N-1)T_s)]^T \), with \( \mathbf{x}_0 = \{\mathbf{u}_0, \mathbf{y}_0\} \) and \( \mathbf{X}_0 = \{\mathbf{U}_0, \mathbf{Y}_0\} \) the (true) DFT vectors containing all the (shifted) DFT components of the input and output spectrum, respectively. The system matrices \( \{\mathbf{A}(\theta_a), \mathbf{B}(\theta_b)\} \in \mathbb{C}^{N \times N} \) have a band structure with a band width of respectively \( \{2n_{ka} + 1, 2n_{kb} + 1\} \), and they are only a function of the Fourier coefficients of, respectively, \( a_n(t) \) (i.e. \( \theta_a \)) and \( b_n(t) \) (i.e. \( \theta_b \)) in (2). The parameter vectors \( \{\theta_a, \theta_b\} \) are defined as \( \theta_x = \text{vec}\{\mathbf{X}^T\} \in \mathbb{C}^{(n_a+1)/(2n_{ka}+1) \times 1} \) with \( x = a, A \) or \( b, B \), where the vec-operator stacks the columns of the matrix on top of each other, and \( \mathbf{X}^T \) is the matrix transpose of \( \mathbf{X} \).

### 2.2 Frozen system

When the dynamic of a system is slowly varying, then the concept of a frozen system makes sense. The term was first introduced by Zadeh in 1950, [21] to analyze time-varying networks. The frozen system has a fictional character and it is the system one obtains by freezing the physical parameters of the time-varying system at a constant time instant, say at \( t = t^* \). This definition has a practical side and fits into the framework of linear parameter varying (LPV) systems [6]. The frozen transfer function (FTF) is then defined as the transfer function of the frozen system; or equivalently, it is the transfer function one obtains by freezing the system parameters \( a_n(t) \) and \( b_n(t) \) in (1) at a constant time instant \( t^* \), viz.

\[
G_f(s, t^*) = \frac{\sum_{n=0}^{n_b} b_n(t^*) s^n}{\sum_{n=0}^{n_a} a_n(t^*) s^n} = \frac{B(s, t^*)}{A(s, t^*)}
\]

with \( s \in \mathbb{C} \) the Laplace variable.

The FTF in (8) can also be factorized in both the numerator and the denominator, viz.

\[
G_f(s, t^*) = K(t^*) \frac{\sum_{n=0}^{n_b} (s - z_n(t^*))}{\sum_{n=0}^{n_a} (s - p_n(t^*))}
\]

with \( K(t^*) \) the frozen gain and with \( z_n(t^*), p_n(t^*) \) respectively the frozen zeroes and the frozen poles. The LPTV system is stable if the poles of the system function (also called the instantaneous transfer function, see Section 2.3 for the definition) lie in the left part of the s-plane, [22], but is not necessarily stable if the frozen poles of the FTF lie in the left portion of the s-plane, (see example 3.22 in [7]). We refer to [4] for a general definition of poles, also termed the Floquet exponents, denoted by \( \lambda_n \) for \( n = \{0, 1, .., n_a\} \), for
LPTV systems. However, if the system is slowly time-varying then the idea of frozen poles and zeroes makes sense and it gives an intuitive insight into the time evolution of the system’s dynamics. Slowly means that the length of the impulse response, $T_{imp}$, is much smaller than the duration of the time-variation, $T_{sys}$.

From the frozen complex poles $p_n(t^*)$ we can then extract the frozen modal parameters like the frozen resonance frequency, $f_n(t^*) = \frac{1}{2\pi} \| p_n(t^*) \|$ ($\| x \|$ is the magnitude of $x$) and the corresponding frozen damping ratio, $\xi_n(t^*) = \frac{1 - \text{Re}(p_n(t^*))}{\| p_n(t^*) \|}$. The frozen time constant, $\tau_n(t^*)$, can be obtained from the frozen real poles as $\tau_n(t^*) = \frac{1}{p_n(t^*)}$.

### 2.3 Frozen transfer function versus instantaneous transfer function

The concept of the FTF is not a general description for a linear (periodically) time-varying system, [21]. A general definition of a transfer function for time-varying systems is defined as the Laplace transform of the time-varying impulse response (or time-varying kernel), denoted as $G(s,t)$, which depends on both the (complex) frequency (i.e. dynamics) and the time instant (i.e. time-variations) and is called the instantaneous transfer function (ITF) in [10], or the system function in [21]. Briefly stated, the ITF can also be seen as being the time-varying response to a complex exponential, $u_0(t) = e^{st}$,

$$u_0(t) = e^{st} \rightarrow y_0(t) = |G(s,t)|e^{st + \mathcal{L}[G_I(s,t)]}. \quad (10)$$

The similarity in (10) with the LTI-response is obvious. The distinction is in the fact that the amplitude change and the phase shift are now time-dependent. Ignoring the time-dependency in $G_I(s,t)$ means that the ITF reduces to the well-known transfer function concept, $G(s)$, for LTI systems. The ITF is thus a generalized concept of a transfer function for time-varying systems and is an invariant of all possible chosen model structures (ODE, state space, impulse response approach, ...).

The FTF at $t^*$, $G_f(s,t^*)$, describes well the true dynamics at $t^*$ if the system parameters remain almost constant over the duration of the impulse response, [21]; while the ITF, $G_I(s,t^*)$, is the true description of the dynamics of the time-varying system at time instant $t^*$ as if we would have made a snapshot of the behavior of the system at $t^*$. For LTI systems there is no confusing which description we have to use, since both, the FTF as well as the ITF, boil down to the well-known concept of a transfer function. For time-varying systems the choice is a bit delicate in some sense. Nevertheless, in [21] it is shown that both concepts (FTF and ITF) tend to approach each other and the speed of time-variation is decreased, and the error one makes at $t^*$ can exactly be calculated as

$$e_G(s,t^*) = G_f(s,t^*) - G_I(s,t^*) = \sum_{n=1}^{n=\infty} \frac{1}{n! A(s,t^*)} \frac{\partial^n A(s,t^*)}{\partial s^n} \frac{\partial^n G_I(s,t^*)}{\partial t^n}. \quad (11)$$

Hence, $G_f(s,t^*)$ can be viewed as a first order approximation of $G_I(s,t^*)$, meaning that $G_f(s,t^*)$ is “less rich” in information than $G_I(s,t^*)$. The “error of variation”, $e_G(s,t^*)$, provides an indication about the speed of time-variation.

### 3 Identification procedure

#### 3.1 Nonparametric noise model

In order to set up an identification scheme noise assumptions are required. As was mentioned in the introduction, a nonparametric noise model will be used, since it circumvents the need of an order selection of the noise model.

The noise disturbances are assumed to have an additive character, viz.

$$\begin{align*}
U(k) &= U_0(k) + N_a(k) \\
Y(k) &= Y_0(k) + N_y(k)
\end{align*} \quad (12)$$
with $N_u(k), N_y(k)$ respectively the DFT of the input and output noise at the $k^{th}$ frequency bin. We are thus working in an errors-in-variables framework where both the output as well as the input are allowed to be corrupted by noise. Furthermore, it is assumed that the noise sources are zero mean, circular complex normally distributed and uncorrelated over the frequency, such that the following holds:

$$
\begin{align*}
\mathbb{E}\{N_u(k)N_u(l)\} &= \sigma_u^2(k) \delta_{kl} \\
\mathbb{E}\{N_y(k)N_y(l)\} &= \sigma_y^2(k) \delta_{kl} \\
\mathbb{E}\{N_y(k)N_u(l)\} &= \sigma_{yu}(k) \delta_{kl}
\end{align*}
$$

with $\delta_{kl}$ the Kronecker symbol (being 1 if $k = l$ and 0 otherwise) and where $\mathbb{E}\{\cdot\}$ stands for the expected value operator ($\vec{X}$ denotes the complex conjugate of $X$). Note that noise coloring is allowed due to the frequency dependency of the noise (co-)variances in (13), $\sigma_x^2(k)$ with $x = \{u, y, yu\}$. The input-output noise disturbances are assumed to be stationary and are thus modeled as filtered, band-limited white noise $E(k)$ [16], viz.

$$
\begin{align*}
N_u(k) &= H_u(j\omega_k)E_u(k) + T_u(j\omega_k) \\
N_y(k) &= H_y(j\omega_k)E_y(k) + T_y(j\omega_k)
\end{align*}
$$

with $T_x(j\omega_k), x = \{u, y\}$ the noise transient (noise leakage), which is a rational form in $j\omega_k$ and decreases towards zero at the rate $O\left(T^{-\frac{1}{2}}\right)$ compared with the noise term $H_x(j\omega_k)E_x(k), x = \{u, y\}$, [16]. Since the noise leakage becomes less important for increasing measurement time, $T$, it will be ignored from now on.

### 3.2 Weighted nonlinear least square estimator

Because the frequency domain model (7) is linear in the (true) input-output spectra, viz.

$$
\mathcal{M}(\theta) Z_0 = \begin{bmatrix} \mathcal{A}(\theta_a) & -\mathcal{B}(\theta_b) \end{bmatrix} \begin{bmatrix} Y_0 \\ U_0 \end{bmatrix} = 0,
$$

with $\theta = [\theta_a^T \theta_b^T]^T$ containing all the Fourier coefficients of the system parameters (2), and, in addition, the noise sources are assumed to be time-invariant, all the results of the Markov estimator derived in chapter 17 in [14] will apply here. As a result, the maximum likelihood (ML) estimator will be set up due to its consistency (the estimated parameters $\hat{\theta}$ converge to the true value $\theta_0$ for a growing amount of data, $T \rightarrow +\infty$) and asymptotic efficiency (it reaches the Cramér-Rao lower bound) property, [14]. Note that for PTV systems consistency is well-defined because the periodicity of the time-variation, $T_{sys}$, is assumed to be of a finite duration.

When noise is present in the measurements $Z$, model (15) will never be zero and residuals are introduced,

$$
e_N(\theta) = \mathcal{M}(\theta) Z \approx 0, \quad e_N(\theta) \in \mathbb{C}^{N\times 1},
$$

even if the model is evaluated in the true parameters $\theta_0$. From an application point of view we are not interested in the whole frequency band, yet only a (small) portion of the frequency band draws our attention, viz.

$$
e(\theta) = S_F e_N(\theta) = S_F \mathcal{M}(\theta) Z \approx 0, \quad e(\theta) \in \mathbb{C}^{F\times 1}.
$$

$S_F$ is formed by the identity matrix $I_N \in \mathbb{R}^{N\times N}$, where the rows that do not correspond with the frequency indices in the frequency band of interest are eliminated. The ML estimator requires the covariance of the vector of residuals, $C_e = \text{cov}\{e\} \in \mathbb{C}^{F\times F}$, which can be computed by means of (12), (13), (15) and (17), viz.

$$
C_e(\theta) = S_F C_{e_N}(\theta) S_F^T
$$

$$
C_{e_N}(\theta) = \mathcal{A}(\theta_a) C_y \mathcal{A}(\theta_a)^H + \mathcal{B}(\theta_b) C_u \mathcal{B}(\theta_b)^H - 2\text{Herm}\{\mathcal{A}(\theta_a) C_{yu} \mathcal{B}(\theta_b)^H\}
$$
with \( C_x, x = \{u, y, y_u\} \) the (cross)-covariance matrix of the noise. These covariance matrices are diagonal matrices due to the noise assumptions in (13). The Hermitian-operator in (19) is defined as \( \text{Herm}(X) = \frac{X + X^H}{2} \) (\( X^H \) is the complex conjugate transpose of \( X \)). Even when the noise sources are stationary, the vector of residuals \( e(\theta) \) is correlated over the frequency (i.e. \( C_e(\theta) \) is not diagonal and has a correlation length of \( \max\{2n_k_a, 2n_k_b\} \)). As \( C_e(\theta) \in \mathbb{C}^{F \times F}, F = \mathcal{O}(N) \) in (18) needs to be inverted during the minimization process and, furthermore, it is expected that the off-diagonals of \( C_e(\theta) \) are small compared with the elements on the main diagonal (i.e. it is a slowly varying system), the following weighted nonlinear least square (WNLS) estimator is then proposed

\[
\hat{\theta}_{\text{WNLS}} = \arg\min_{\theta} e(\theta)^H \text{diag}\{C_e(\theta)\}^{-1}e(\theta), \text{ subject to } \left\{ \begin{array}{l}
\|\theta_R\|_2 = 1, \\
\theta = C\theta_R
\end{array} \right. \tag{20}
\]

where \( \theta_R \in \mathbb{R}^{n_R \times 1} \) are the (real) independent parameters which should be estimated from the available measurements \( Z \) and with \( C \in \mathbb{C}^{n_R \times n_R} \) the complex constraint matrix (as will elaborated further on). \( \text{diag}\{X\} \) is a diagonal matrix made of only the main diagonal elements of \( X \). It is clear that, since we neglect the off-diagonals in \( C_e(\theta) \), some efficiency will be lost. As it is assumed that the system is slowly varying the contributions of the off-diagonals are becoming less important (in the case of LTI no off-diagonals in \( C_e(\theta) \) are present under the assumptions (12) and (13)), and we expect a (very) small efficiency loss. Moreover, it can be proven that the WNLS estimator is a consistent estimator. The reason why the WNLS estimator is preferred over the ML estimator is that it avoids inverting the (band) square matrix \( C_e \in \mathbb{C}^{F \times F} \) in (18), which grows with the amount of frequency domain data.

### 3.2.1 Constraints on the system parameters

As the Fourier coefficients of the system parameters in (2) (i.e. the components in \( \theta_a \) and \( \theta_b \)) are not independent of each other, constraints have to be enforced on the complex periodic coefficients \( A_{n,k}, B_{n,k} \in \mathbb{C} \) in (2) by a constraint real matrix \( T \) to guarantee that the estimated time-dependent parameters \( a_n(t), b_n(t) \in \mathbb{R} \) in (1) are real. For that reason, the following constraints are imposed on \( A_{n,k}, B_{n,k} \):

\[
\begin{align*}
\text{Re}\{A_{n,k}\} &= \text{Re}\{A_{m,-k}\} \\
\text{Im}\{A_{n,k}\} &= -\text{Im}\{A_{m,-k}\} \\
\text{Im}\{A_{n,0}\} &= 0 \\
\text{for } n &= \{0, 1, ..., n_a\} \quad \text{for } n \in \{0, 1, ..., n_b\} \\
\text{for } k &= \{-n_{k_a}, -n_{k_a} - 1, ..., n_{k_a}\} \quad \text{for } k \in \{-n_{k_b}, -n_{k_b} - 1, ..., n_{k_b}\}
\end{align*}
\tag{21}
\]

with \( \text{Re}\{x\} \) and \( \text{Im}\{x\} \) respectively the real and imaginary part of \( x \). In order to apply the constraint matrix \( T \) on \( \theta = [\theta_a^T, \theta_b^T]^T \in \mathbb{C}^{n_R \times 1} \), \( \theta \) ought to be first moulded in its real version, viz.

\[
\theta_{re} = \begin{bmatrix} \text{Re}\{\theta\} \\ \text{Im}\{\theta\} \end{bmatrix}.
\tag{22}
\]

The constraints (21) are linear-in-the-parameters and, therefore, they can be written as a single matrix equation

\[
\theta_{re} = T \theta_R
\tag{23}
\]

with \( T \in \mathbb{R}^{n_R \times n_R} \) a sparse matrix consisting of \( \{0, 1, -1\} \) and \( \theta_R \in \mathbb{R}^{n_R \times 1} \) the (real) independent parameters which should be estimated from the available measurements \( Z \).

### 3.2.2 Minimization algorithm with constraints

The WNLS costfunction (20) is minimized subject to the constraints (21) by making use of a Newton-Gauss algorithm. The convergence region of the Newton-Gauss method can be expanded in practice using a Levenberg-Marquardt algorithm.
This algorithm requires the calculation of the (real) Jacobian \( J_R \in \mathbb{R}^{2F \times n_{\theta R}} \)
\[
J_R[\cdot,\cdot] = \frac{\partial \varepsilon_{re}(k, \theta)}{\partial \theta_R} = \frac{\partial \varepsilon_{re}(k, \theta)}{\partial \theta_{re}} T, \quad J_R[\cdot,\cdot] \in \mathbb{R}^{2 \times n_{\theta R}}
\] (24)
with \( X[\cdot,\cdot] \) the \( k^{th} \) row of \( X \) and where \( \varepsilon(k, \theta) = \frac{e(k, \theta)}{\sigma_e(k, \theta)} = \frac{e_k(\theta)}{\sqrt{\text{diag} \{ C_0(\theta) \}}[k,k]} \) are the normalized residuals and \( x_{re} \) takes the real and imaginary parts of \( x \) and stacks them on top of each other (see (22)).

The cost function (20) being nonlinear in \( \theta \) (or \( \theta_R \)), its minimization requires an initial estimate, \( \hat{\theta}_R^{(0)} \), which is given by the total least square estimator, [9]. An update of the estimate \( \hat{\theta}_R^{(i)} \) is then calculated from the previous iteration step as
\[
\hat{\theta}_R^{(i)} = \hat{\theta}_R^{(i-1)} + \delta \hat{\theta}_R^{(i)}
\] (25)
where \( \delta \hat{\theta}_R^{(i)} \) is the solution of
\[
\left[ \begin{array}{c} J_R(\hat{\theta}_R^{(i-1)}) \\ \lambda I_{n_{\theta R}} \end{array} \right] \delta \hat{\theta}_R^{(i)} = - \left[ \begin{array}{c} \varepsilon_{re}(\hat{\theta}_R^{(i-1)}) \\ 0_{n_{\theta R}} \end{array} \right],
\] (26)
which can be computed from the singular value decomposition (SVD) of \( J_R(\hat{\theta}_R^{(i-1)}) \) and where \( \varepsilon(\theta_R) \) is a column vector whose \( k^{th} \) element is \( \varepsilon(k, \theta_R) = \frac{e_k(\theta_R)}{\sigma_e(k, \theta_R)} \). An initial value for \( \lambda \) in (26) could be for e.g. \( \lambda = \frac{\sigma_1}{\text{MM}} \) where \( \sigma_1 \) is the largest singular value of \( J_R(\hat{\theta}_R^{(0)}) \). An estimate of \( \theta \) follows then easily from (22) and (23), viz.
\[
\hat{\theta} = C \theta_R = R T \hat{\theta}_R \quad \text{with} \quad R = [I_{n_{\theta}} \sqrt{-1} I_{n_{\theta}}] \in \mathbb{C}^{n_{\theta} \times 2n_{\theta}},
\] (27)
where \( I_{n_{\theta}} \in \mathbb{R}_{n_{\theta} \times n_{\theta}} \) is the identity matrix. Finally, an estimate of the system parameters (i.e. \( \hat{a}_n(t), \hat{b}_n(t) \) in (1)) is then obtained from the Fourier coefficients stored in \( \theta = [\hat{\theta}_d^T \hat{\theta}_b^T]^T \), as given in (2).

### 3.2.3 Model validation

In the last stage of the identification process the estimated model should be validated. Does the estimated model describe the noisy data or are there unmodeled dynamics? What is the quality of the estimated model? Can modeling errors be detected? To answer these questions validation test tools must be developed. In this paper two tests will be used: (i) the test on the cost function and (ii) the whiteness test on the residuals.

1) Test on the WNLS cost function
A useful tool to determine whether or not modeling errors are present in the estimated model is to evaluate the cost function in the minimized parameters and to check whether the cost function lies within its 95\% confidence interval. If not, then it is likely that model errors are present. Hence, in the absence of model errors, the expected value and the variance of the WNLS cost function in the minimized parameters for deterministic \( Z_0 \) (proven in chapter 17 of [14]) become
\[
\mathbb{E}\left\{ V_{\text{WNLS}}(\hat{\theta}_R, Z) \right\} \approx V_{\text{WNLS}}(\theta_0, Z_0) + F - \frac{(n_{\theta_R} - 1)}{2} = F - \frac{(n_{\theta_R} - 1)}{2}
\] (28)
\[
\text{var}\{ V_{\text{WNLS}}(\hat{\theta}_R, Z) \} \approx 2 V_{\text{WNLS}}(\theta_0, Z_0) + F - \frac{(n_{\theta_R} - 1)}{2} = F - \frac{(n_{\theta_R} - 1)}{2}
\] (29)
with \( n_{\theta_R} - 1 \) the number of identifiable model parameters (free parameters). Based on formulas (28) and (29) the 95\% confidence bound can be constructed. A too low value of the WNLS cost function indicates typically an over-estimation of the (co)-variances in (13).
2) Test on the normalized residuals $\varepsilon$

A second method of model validation consists of inspecting the distribution of the residuals for a growing amount of data. In ([14], chapter 17) it is shown that for the Markov estimator (i.e. the maximum likelihood estimator) the residuals are asymptotically white (i.e. $\text{cov}\{\varepsilon(\hat{\theta}_R)\} \approx \text{identity matrix}$). Since in practice the off-diagonals of $\mathbf{C}_e(\theta)$ in (18) can be omitted for slow variations, the results are also applicable here. Therefore, the residuals, $e(k, \hat{\theta}_R)$, and the standard deviation of the residuals, $\sigma_e(k, \hat{\theta}_R)$, evaluated in the estimated parameters $\hat{\theta}_R$, should lie in the neighborhood of each other. The $\text{prob}\%$-confidence bound of $e(k, \hat{\theta}_R)$, where $e(k, \hat{\theta}_R)$ is asymptotically zero mean, circular complex, normally distributed, is given by $\sqrt{-\log(1-\text{prob})}\sigma_e(k, \hat{\theta}_R)$ (see formula (15) in [13]).

### 3.3 Estimation of a nonparametric noise model

As shown in the previous section, $\sigma_e^2(k, \hat{\theta}_R) = \text{diag}\{\mathbf{C}_e(\hat{\theta}_R)\}_k$ in (18) requires the knowledge of the noise (co)-variances (13) as a function of the frequency. In practice, these (cross-) auto-power spectra of the noise are not known and have to be estimated from the available data $\mathbf{Z}$. A nonparametric estimate of the noise power spectrum as a function of the frequency can be obtained as follows. Assume that we can observe $P \geq 2$ periods of the input-output signals,

$$T_{\text{meas}} = PT = qT_{\text{exc}} = pT_{\text{sys}} = NT_s, \{P, q, p, N\} \in \mathbb{N}_0.$$  

where $T_{\text{meas}}$ is the total measurement time and $T$ the periodicity of the output signal, $y_0(t+T) = y(t)$, then a nonparametric estimate of the noise (co)-variances can be calculated using the following strategy:

1. Divide the entire measurement record, $N_{\text{meas}}$ samples, in $P$ subrecords with $N$ samples per subrecord.

2. Calculate the input-output DFT spectra (5) for each subrecord in the frequency band of interest $\mathbf{U}^{[i]}(k)$, $\mathbf{Y}^{[i]}(k)$, $i = \{1, ..., P\}$ and $k = \{1, ..., F\}$.

3. The sample noise (co-)variances are then computed directly from these DFT spectra as:

$$\hat{\sigma}_u^2(k) = \frac{1}{P-1} \sum_{i=1}^P \left| \mathbf{U}^{[i]}(k) - \hat{\mathbf{U}}(k) \right|^2 \quad \text{with} \quad \hat{\mathbf{U}}(k) = \frac{1}{P} \sum_{i=1}^P \mathbf{U}^{[i]}(k),$$  

$$\hat{\sigma}_y^2(k) = \frac{1}{P-1} \sum_{i=1}^P \left| \mathbf{Y}^{[i]}(k) - \hat{\mathbf{Y}}(k) \right|^2 \quad \text{with} \quad \hat{\mathbf{Y}}(k) = \frac{1}{P} \sum_{i=1}^P \mathbf{Y}^{[i]}(k),$$

$$\hat{\sigma}_{yu}^2(k) = \frac{1}{P-1} \sum_{i=1}^P \left( \mathbf{Y}^{[i]}(k) - \hat{\mathbf{Y}}(k) \right) \left( \mathbf{U}^{[i]}(k) - \hat{\mathbf{U}}(k) \right).$$

If the nonparametric noise model is calculated with formulas (31), (32) and (33) then the test on the WNLS cost function from Section 3.2.3, given by the expressions (28) and (29), should be adapted to

$$\mathbb{E}\left\{V_{\text{WNLS}}(\hat{\theta}_R, \mathbf{Z})\right\} \approx \frac{P-1}{P-2} \left( F - \frac{(n\theta_R - 1)}{2} \right),$$

$$\text{var}\{V_{\text{WNLS}}(\hat{\theta}_R, \mathbf{Z})\} \approx \frac{(P-1)^3}{(P-2)^3(P-3)} \left( F - \frac{(n\theta_R - 1)}{2} \right), P > 7.$$  

For the proof we refer to chapter 8 in [14].
4 Uncertainty analysis

In Section 3 it was shown how consistent estimators can be obtained for LPTV systems from noisy input-output observations. Since a stochastic estimate without uncertainty bounds loses its value, the variability of the estimated quantities should be computed. Recall that any model-related quantity, $f(\hat{\theta}_R)$, (e.g., frozen transfer function, frozen modal parameters, ...) can be calculated from the covariance of the estimated parameters with the following transformation formula

$$\text{cov}\{f(\hat{\theta}_R)\} \approx \left. \frac{\partial f(\theta_R)}{\partial \theta_R} \right|_{\theta_R=\hat{\theta}_R} \text{cov}\{\hat{\theta}_R\} \left( \left. \frac{\partial f(\theta_R)}{\partial \theta_R} \right|_{\theta_R=\hat{\theta}_R} \right)^H. \quad (36)$$

The transformation formula (36) is exact if the relationship between $f(\hat{\theta}_R)$ and $\theta_R$ is linear, (i.e., $f(\hat{\theta}_R) = K_{\theta_R} \hat{\theta}_R$, with $K_{\theta_R}$ a constant matrix with appropriate dimensions).

In order to calculate uncertainties on the frozen transfer function (FTF) and on the frozen modal parameters, like the frozen poles, frozen resonance frequency, frozen damping ratio, and so on; the covariance of the system parameters, $\text{cov}\{\hat{\theta}_R\}$, must be known. As already mentioned in the previous section we are dealing with LPTV systems with slow dynamic variations, such that the results of the Markov estimator (see chapter 17 in [14]) are valid. An approximate expression for the covariance of the parameters is then given by means of the Jacobian, as

$$\text{cov}\{\hat{\theta}_R\} \approx \left( 2I_R(\hat{\theta}_R)J_R(\hat{\theta}_R) \right)^{-1}. \quad (37)$$

From (37) it is possible to obtain a covariance expression for, respectively, the estimate of the $a_n(t)$ and $b_n(t)$ system parameters in (1) by making use of transformation formula (36).

We have seen in Section (2.2) that the frozen transfer function (FTF) is a useful concept to describe the evolution of the frozen dynamics of LPTV systems. Therefore, once the FTF is estimated from (8), uncertainty ellipsoids of the parametric FTF have to be constructed. In order to calculate these ellipsoids the following uncertainty quantities of the FTF are needed, viz. by using (36) we have

$$\mathbb{E} \left\{ \left| \Delta \hat{G}_f(s, t^*) \right|^2 \right\} = \sigma^2_{\hat{G}_f}(s, t^*) = \text{var}\{\hat{G}_f(s, t^*)\} \approx \left. \frac{\partial G_f(s, t^*)}{\partial \theta_R} \right|_{\theta_R=\hat{\theta}_R} \text{cov}\{\hat{\theta}_R\} \left( \left. \frac{\partial G_f(s, t^*)}{\partial \theta_R} \right|_{\theta_R=\hat{\theta}_R} \right)^H, \quad (38)$$

$$\mathbb{E} \left\{ \Delta \hat{G}_f^2(s, t^*) \right\} \approx \left. \frac{\partial G_f(s, t^*)}{\partial \theta_R} \right|_{\theta_R=\hat{\theta}_R} \text{cov}\{\hat{\theta}_R\} \left( \left. \frac{\partial G_f(s, t^*)}{\partial \theta_R} \right|_{\theta_R=\hat{\theta}_R} \right)^T. \quad (39)$$

Expression (39) is necessary since $\hat{G}_f(s, t^*)$ is in general not circular complex distributed. Note that the uncertainty on the FTF, $\sigma^2_{\hat{G}_f}(s, t^*)$, in (38) is a 2-dimensional function that depends both on the (complex) frequency as well as on the frozen time instant.

Sometimes we are interested in the uncertainty of other quantities, for instance, of the estimated frozen poles, of the frozen modal parameters, etc. This is, for example, the case in applications where modal analysis is suited, [12]. Note that at a constant time instant $t^*$ the uncertainty of the estimated frozen poles can be processed as if it was time-invariant at $t^*$. Therefore, the results in [12] can be used, which are derived for LTI systems. We refer to [12] for a detailed explanation of the elaboration of the uncertainty analysis of any modal parameter (poles, resonance frequency, mode shape, damping, ...). To obtain a variance expression for the estimated frozen poles, $\sigma^2_{\hat{p}_n}(t^*) = \text{var}\{\hat{p}_n(t^*)\}$, the estimated frozen resonance frequency, $\sigma^2_{\hat{f}_n}(t^*) = \text{var}\{\hat{f}_n(t^*)\}$, the estimated frozen damping ratio, $\sigma^2_{\hat{c}_n}(t^*) = \text{var}\{\hat{c}_n(t^*)\}$ and the estimated frozen time constant, $\sigma^2_{\hat{\tau}_n}(t^*) = \text{var}\{\hat{\tau}_n(t^*)\}$, the covariance matrix of the $\mathbf{a}$-parameters, $\text{cov}\{\hat{\mathbf{a}}(t^*)\}$, is needed, which is can readily elaborated elaborated using the transformation formula (36) in the linear case.
5 Experimental results on a flexible robot arm

The identification procedure in Section 3 has been applied to real data on a mechanical system with varying length of the robot arm. The device under test is a mechanical robot arm with two degrees of freedom (see Fig. 1 (left) for the experimental set up). The two axes are driven by two linear motors. When the extendible robot arm (scheduling in Fig. 1) is changed, the stiffness and the resonance frequency of the system will also vary according to the laws of physics. Hence, the XY-table becomes time-varying and we need identification tools for time-varying systems – developed in Section 3 for (linear) PTV systems – to identify such systems.

![Experimental set up of the XY-table with its corresponding axes. The input, output and scheduling parameter are also depicted.](image)

For the PTV experiment, a full multisine signal (3) in the frequency band \([80.05, 150]\) Hz with \(F = 1400\) harmonics, sampled at a rate of \(f_s = 2\) kHz, was applied as a current reference of the linear motor of the X-axis. The output of the considered PTV system is the acceleration of the small mass at the tip of the extendible robot arm (see Fig. 1 (left)). In addition, \(P = 11\) periods of the input-output signal were gathered, giving a total measurement time \(T_{\text{meas}} = PT = 220\) s (\(\{T_{\text{meas}}, T\}\) are defined in (30)). We only focussed on the second resonance of the system, since it turned out that at this resonance the degree of time-variation was the highest.

As mentioned in Section 2.1 it is desirable to perform first a time-invariant (TI) experiment (i.e. constant scheduling in Fig. 1 (left)) in order to obtain the system orders \(n_a\) and \(n_b\) in (1) for the PTV model using well-established identification methods for LTI systems, [14]. Therefore, a TI experiment was carried out in the frequency band \([80.05, 140]\) Hz and the robot arm was excited with a full multisine signal (3) with \(F = 1200\) harmonics, which was sampled at a rate of \(f_s = 2\) kHz. The value of the constant scheduling has been chosen somewhere in the range of the periodic trajectory of the scheduling in the PTV experiment (see the right plot in Fig. 2). An in-depth analysis of the input-output data have shown that the TI system behaved to some extent nonlinearly. The results of the TI experiment are summarized in Fig. 1 (right) by comparing the parametric frequency response function (FRF), obtained with the scheme of Section 3 (with \(n_{k_c} = n_{k_b} = 0\) in (2) and weighted with the nonlinear distortion instead of the noise in the WNLS cost function), with the nonparametric FRF (obtained from the method in [15]). We have made use of the nonparametric methods, described thoroughly in [15], to get an idea of the level of the nonlinear distortions in the TI experiment. In Fig. 1 (right) it can be seen that the discrepancy between the parametric FRF
and the nonparametric one is scattered around the (variance of the) nonlinear distortion, which validates the estimated 5/8 model \((n_b = 5\) and \(n_a = 8\)).

Figure 2: Left: 3D-view of the estimated parametric FTF, \(G_f(j\omega, t^*)\), of the XY-table. Right: The measured scheduling parameter (length of the extendible robot arm in Fig. 1) as a (periodic) function of time.

In order to introduce time-variations in the PTV experiment, the length of the extendible robot arm (Y-axis) was changed by imposing a multisine signal with 3 harmonics with \(T_{sys} = 2.5\) s, \(p = \frac{T}{T_{sys}} = 8\), \(q = 1\), where \(\{p, q\}\) are defined in (4) (see right plot in Fig. 2 for the evolution of the scheduling). After collecting the input-output data, the parameters of model (1) are estimated in the frequency domain using the methodology in Section 3 yielding the following model orders \(\{n_a = 8, n_{k_a} = 3, n_b = 5, n_{k_b} = 0\}\). From these estimates, the FTF (8) is obtained. A 3D-view of the estimated FTF is depicted in Fig. 2. The following observations can be made:

- The resonance frequency is varying between 100 – 130 Hz (see also Fig. 3).
- The shape of the imposed scheduling parameter is reflected in the resonance frequency (compare left and right plot in Fig. 2 and see also Fig. 3).

To have also an idea about the evolution of the frozen modal parameters of the XY-table, we have depicted in Fig. 3 the frozen resonance frequency and the frozen damping ratio together with their 95% confidence bounds. The (frozen) damping ratio has a relatively high uncertainty since it is a measure of the angle between the location of the estimated frozen poles and the imaginary axis.

In order not to overload the figures and to ease the comparison between different quantities, the Power Average over Time (PAoT) of quantity \(x(j\omega_k, t)\) is computed and is defined as

\[
\text{PAoT}\{x(j\omega_k, t)\} = \sqrt{\frac{1}{T_{sys}} \int_0^{T_{sys}} |x(j\omega_k, t)|^2 dt} \approx \sqrt{\frac{1}{N_t} \sum_{i=0}^{N_t-1} |x(j\omega_k, i\delta t)|^2}
\] (40)

with \(T_{sys} = N_t\delta t\) and \(\delta t\) the integration step. The integration error can be made arbitrary small (\(\delta t\) can be chosen arbitrary small) since \(x(j\omega_k, t)\) is known parametrically in the time variable \(t\).

The dynamic of the XY-table is varying fast to some extent. This confirmed in Fig. 4 by the PAoT of the estimated error \(\hat{e}_G(j\omega_k, t^*)\) in (8) (difference between the estimated FTF and the estimated ITF). In Fig. 4, the PAoT of \(\hat{e}_G(j\omega_k, t^*)\) is relatively large at the resonance frequencies compared to the level of the FTF. Hence, it is meaningful to differentiate between both concepts (FTF and ITF), conform the clarification in
Section 2.3. It is then to the users to decide which definition suits best their application. The fast varying behavior was expected due to the large amplitude of the scheduling parameter that was applied (see Fig. 2) compared with the length of the robot arm, and the short periodicity duration of the scheduling parameter \( T_{sys} = 2.5 \) s compared with the periodicity of the output-signal, \( T = 20 \) s, \( p = \frac{T}{T_{sys}} = 8 >> 1 \).

Figure 3: Left: The estimated frozen resonance frequency as a function of the frozen time instants within \([0, T_{sys}]\) (light grey), while the 95\% uncertainty bound \([\hat{f}(t^*) - 2\sigma_f(t^*), \hat{f}(t^*) + 2\sigma_f(t^*)]\), \( t^* \in [0, T_{sys}] \) is depicted in black. Right: The estimated damping ratio as a function of frozen time instants within \([0, T_{sys}]\) (light grey), while the 95\% uncertainty bound \([\hat{\xi}(t^*) - 2\sigma_\xi(t^*), \hat{\xi}(t^*) + 2\sigma_\xi(t^*)]\), \( t^* \in [0, T_{sys}] \) is depicted in black.

Figure 4: The estimated FTF at some frozen time instants, \( t^* \in [0, 2.5] \) s (black). The PAoT of the difference between the estimated FTF and the estimated ITF is given by the bold grey line, while the PAoT of the estimated error computed with (11) is depicted in thin white. The PAoT of the uncertainty of the parametric FTF, \( \sigma_{G_f}(j\omega_k, t^*) \) in (38), is given in black dashed line.

It should be mentioned that we have used the (variance of the) output nonlinear distortion level as weighting in the WNLS cost function (20) and not the (variance of the) noise (i.e. \( C_y = C_y^{NL}, C_u = C_u^{noise} \) and \( C_{yu} = C_{yu}^{noise} \) in (19)), where the variance of the output nonlinear distortion, \( C_y^{NL}[k,k] \) is estimated with the method described in [10]. Only output nonlinear distortions were taken into account because no significant input nonlinear distortions could be detected, therefore, the output-error framework in [10] is applicable here.

Finally, an additional validation test was done on the WNLS cost function (20) and the normalized residuals,
as explained in Section 3.2.3. It was found that the value of the cost function lies within its 95% confidence interval ([1313, 1520]) and that 5.8% of the residuals was found outside the 95% confidence region, which validates the estimated model for the XY-table.

6 Conclusions

A simplified version of the maximum likelihood estimator (i.e. WNLS estimator) for identifying linear periodically time-varying systems has been developed in an errors-in-variables environment. The WNLS estimator turned out to be consistent and for slowly varying systems a “small” efficiency loss was expected. The important assumptions that were made, for constructing the estimator, are:

- The system can be excited by a band-limited, broad-band periodic signal with a user-defined amplitude spectrum.
- The periodicity of the excitation can be synchronized with that of the time-variation.
- The system parameters $a_n(t)$ and $b_n(t)$ in (1) are band-limited.
- The noise sources are supposed to be stationary, and are modeled as a time-invariant filtered white noise process.

In addition, two concepts of a transfer function for LPTV systems have been introduced, both of which can be obtained from the ODE model. By monitoring the difference between both concepts, also called the error of variation, $e_G(j\omega_k, t*)$, we can quantify the speed of variation in the PTV experiment. Moreover, uncertainty expressions are given on any frozen model-related quantity (as on the frozen transfer function, on the frozen poles, on the frozen resonance frequency, ...). It is also shown in practice how to validate the estimated model via validation test tools on the cost function and the residuals. In this work, a balance has been sought between the provided theoretical tools and their relevance in practice. The developed theory was in good agreement with the measurement results.

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